Quantum information theory (MAT4430) **Spring 2021**

Lecture 18: The decoupling method and the LSD theorem

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In this lecture, we will prove the decoupling theorem and use it to obtain coding schemes for entanglement generation over a quantum channel T.

1 The coherent information and the LSD theorem

We will start with a definition:

Definition 1.1 (Coherent information). *For a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and a quantum state $\sigma \in D(\mathcal{H}_A)$ we define the* coherent information of σ through T *as*

$$I_c(\sigma; T) = H(T(\sigma)) - H\left(\left(\mathrm{id}_{B(\mathcal{H}_A)} \otimes T\right)\left(\mathrm{vec}\left(\sqrt{\sigma}\right)\mathrm{vec}\left(\sqrt{\sigma}\right)^\dagger\right)\right),$$

and the maximum coherent information of T *as*

$$I_c(T) = \max_{\sigma \in D(\mathcal{H}_A)} I_c(\sigma; T).$$

Recall that quantum channels $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and $S : B(\mathcal{H}_A) \to B(\mathcal{H}_E)$ for complex Euclidean spaces $\mathcal{H}_A, \mathcal{H}_B$ and $\mathcal{H}_E$ are *complementary* if there exists an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$T(X) = \mathrm{Tr}_E\left[VXV^\dagger\right] \quad \text{and} \quad S(X) = \mathrm{Tr}_B\left[VXV^\dagger\right].$$

We will sometimes denote by T^c a (standard) complementary channel obtained from a Stinespring dilation. The following lemma gives a useful alternative form of the coherent information in terms of complementary channels:

Lemma 1.2. *For any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and any quantum state $\sigma \in D(\mathcal{H}_A)$ we have*

$$I_c(\sigma; T) = H(T(\sigma)) - H(S(\sigma)),$$

for any quantum channel $S : B(\mathcal{H}_A) \to B(\mathcal{H}_E)$ complementary to T.

Proof. Consider an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$T(X) = \mathrm{Tr}_E\left[VXV^\dagger\right] \quad \text{and} \quad S(X) = \mathrm{Tr}_B\left[VXV^\dagger\right],$$

and define the vector

$$|v_{ABE}\rangle = (\mathbb{1}_A \otimes V)\,\mathrm{vec}\left(\sqrt{\sigma}\right) \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E.$$

Note that

$$\begin{aligned} H\left(\left(\mathrm{id}_{B(\mathcal{H}_A)} \otimes T\right)\left(\mathrm{vec}\left(\sqrt{\sigma}\right)\mathrm{vec}\left(\sqrt{\sigma}\right)^\dagger\right)\right) &= H\left(\mathrm{Tr}_E\left[|v_{ABE}\rangle\langle v_{ABE}|\right]\right) \\ &= H\left(\mathrm{Tr}_{AB}\left[|v_{ABE}\rangle\langle v_{ABE}|\right]\right) = H(S(\sigma)), \end{aligned}$$

where we used that $|v_{ABE}\rangle$ is a pure state and that reduced density operators of the same pure state have the same spectra. The statement of the lemma follows by inserting the previous equation into the definition of $I_c(\sigma; T)$. □

The previous lemma allows a neat physical interpretation of the coherent information of σ through the quantum channel T: The coherent information quantifies how much more information about the input state at A arrives at the receiver B compared to the environment system E. Intuitively, it makes sense that this should be related to the task of quantum communication since sending a pure state with low error means that the environment system E can only be very weakly correlated with the receiving system B and hence has almost no information about the state that was send. We will see in the next sections, that this intuition can be used to prove the following capacity theorem:

Theorem 1.3 (Lloyd, Shor, Devetak)**.** *For any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ we have*

$$Q(T) = \lim_{k\to\infty} \frac{1}{k} I_c\left(T^{\otimes k}\right).$$

We will spend the rest of this lecture proving this result.

2 The decoupling approach

To prove the capacity theorem, we can focus on the task of entanglement generation for which the achievable rates coincide with the achievable communication rates for quantum information (see previous lecture). To find efficient entanglement generation schemes we will apply the so-called *decoupling approach.* The main idea behind this strategy is that a decoder generating entanglement for some given pure input $|\phi_{RA}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_A$ state can be directly obtained from Uhlmann's theorem. The final error of this entanglement generation scheme is given by the distance of the RE marginal of the joined quantum state, obtained from sending the A system of $|\phi_{RA}\rangle$ through the Stinespring isometry of the quantum channel under consideration, from a product state. Intuitively, this says that if after the application of the quantum channel the reference system R and the environment system E are approximately decoupled, then we can approximately generate entanglement between R and the output system B of the channel (see Figure ....). The next theorem makes this intuition precise:

Theorem 2.1 (Decoupling implies code)**.** *Consider a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ with Stinespring isometry $V : \mathcal{H}_A \to \mathcal{H}_E \otimes \mathcal{H}_B$ and a pure state $|\phi_{RA}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_A$ for some complex Euclidean space $\mathcal{H}_R$. For the pure state*

$$|\psi^{REB}\rangle = (\mathbb{1}_R \otimes V)|\phi_{RA}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_E \otimes \mathcal{H}_B$$

we assume that

$$\| \operatorname{Tr}_B \left[|\psi^{REB}\rangle\langle\psi^{REB}|\right] - \frac{\mathbb{1}_R}{\dim(\mathcal{H}_R)} \otimes \tau^E\|_1 \leq \epsilon,$$

for some $\tau^E \in D(\mathcal{H}_E)$. Then, there exists a quantum channel $D : B(\mathcal{H}_B) \to B(\mathcal{H}_R)$ satisfying

$$F(\omega_R, (\mathrm{id}_R \otimes D \circ T)\left(|\phi^{RA}\rangle\langle\phi^{RA}|\right) \geq 1 - \frac{1}{2}\epsilon.$$

Proof. Exercises. □

The decoupling theorem concerns the following situation:

To find coding schemes to share entanglement (and hence to communicate quantum information) we will need the following theorem:

Theorem 2.2 (The decoupling theorem)**.** *Consider complex Euclidean spaces $\mathcal{H}_A$ and $\mathcal{H}_E$, and a (not necessarily normalized) vector $|\phi^{AEB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_E \otimes \mathcal{H}_B$. Furthermore, consider* *a projection $P : \mathcal{H}_A \to \mathcal{H}_A$ such that $\mathcal{H}_R = Im(P) \subset \mathcal{H}_A$ and define the (unnormalized) vectors*

$$|\psi_U^{REB}\rangle = \sqrt{\frac{\dim(\mathcal{H}_A)}{\dim(\mathcal{H}_R)}} \left(PU \otimes \mathbb{1}_E \otimes \mathbb{1}_B\right) |\phi^{AEB}\rangle,$$

for each unitary $U \in \mathcal{U}(\mathcal{H}_A)$. Then, we have

$$\int_{\mathcal{U}(\mathcal{H}_A)} \left\| \operatorname{Tr}_B \left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}| \right] - \frac{P}{\dim(\mathcal{H}_R)} \otimes \tau^E \right\|_1 d\eta(U) \leq \sqrt{\dim(\mathcal{H}_R) \dim(\mathcal{H}_E) \operatorname{Tr}\left[(\tau^{AE})^2 \right]},$$

where we used the reduced operators

$$\tau^E = \operatorname{Tr}_{AB} \left[|\phi^{AEB}\rangle\langle\phi^{AEB}| \right]$$

and

$$\tau^{AE} = \operatorname{Tr}_B \left[|\phi^{AEB}\rangle\langle\phi^{AEB}| \right].$$

We will need a few properties of Haar-integrals:

- By a result from an earlier lecture, we have

$$\int_{\mathcal{U}(\mathcal{H})} UXU^\dagger d\eta(U) = \operatorname{Tr}[X] \frac{\mathbb{1}_\mathcal{H}}{\dim(\mathcal{H})}.$$

- We also need the formula for the UU-twirling channel $T_{UU} : B(\mathcal{H} \otimes \mathcal{H}) \to B(\mathcal{H} \otimes \mathcal{H})$ given by

$$\begin{aligned} T_{UU}(X) &= \int_{\mathcal{U}(\mathcal{H})} (U \otimes U) X (U \otimes U)^\dagger d\eta(U) \\ &= \langle P_{sym}, X\rangle_{HS} \frac{P_{sym}}{\operatorname{Tr}[P_{sym}]} + \langle P_{asym}, X\rangle_{HS} \frac{P_{asym}}{\operatorname{Tr}[P_{asym}]}, \end{aligned}$$

  for

$$P_{sym} = \frac{1}{2}\left(\mathbb{1}_\mathcal{H} \otimes \mathbb{1}_\mathcal{H} + \mathbb{F}\right) \quad \text{and} \quad P_{asym} = \frac{1}{2}\left(\mathbb{1}_\mathcal{H} \otimes \mathbb{1}_\mathcal{H} - \mathbb{F}\right).$$

We can now prove the decoupling theorem:

Proof. Note first that

$$\begin{aligned} &\int_{\mathcal{U}(\mathcal{H}_A)} |\psi_U^{REB}\rangle\langle\psi_U^{REB}| d\eta(U) \\ &= \frac{\dim(\mathcal{H}_A)}{\dim(\mathcal{H}_R)} \int_{\mathcal{U}(\mathcal{H}_A)} \left(PU \otimes \mathbb{1}_E \otimes \mathbb{1}_B\right) |\phi^{AEB}\rangle\langle\phi^{AEB}| \left(U^\dagger P \otimes \mathbb{1}_E \otimes \mathbb{1}_B\right) d\eta(U) \\ &= \frac{P}{\dim(\mathcal{H}_R)} \otimes \tau^{EB}, \end{aligned}$$

where

$$\tau^{EB} = \operatorname{Tr}_A \left[|\phi^{AEB}\rangle\langle\phi^{AEB}| \right].$$

As a consequence, we find that

$$
\begin{aligned}
\int_{\mathcal{U}(\mathcal{H}_A)} &\Big\| \operatorname{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right] - \frac{P}{\dim(\mathcal{H}_R)} \otimes \tau^E \Big\|_2^2 d\eta(U) \\
&= \int_{\mathcal{U}(\mathcal{H}_A)} \langle \operatorname{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right], \operatorname{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right]\rangle \\
&\qquad - \langle \frac{P}{\dim(\mathcal{H}_R)} \otimes \tau^E, \operatorname{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right]\rangle \\
&\qquad - \langle \operatorname{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right], \frac{P}{\dim(\mathcal{H}_R)} \otimes \tau^E\rangle \\
&\qquad + \langle \frac{P}{\dim(\mathcal{H}_R)} \otimes \tau^E, \frac{P}{\dim(\mathcal{H}_R)} \otimes \tau^E\rangle d\eta(U) \\
&= \int_{\mathcal{U}(\mathcal{H}_A)} \operatorname{Tr}\left[\operatorname{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right]^2\right] d\eta(U) - \frac{1}{\dim(\mathcal{H}_R)} \operatorname{Tr}\left[\left(\tau^E\right)^2\right].
\end{aligned}
$$

Let us focus on the first term in the last line. Note that

$$
\operatorname{Tr}\left[\left(\rho^{AE}\right)^2\right] = \operatorname{Tr}\left[(\mathbb{F}_A \otimes \mathbb{F}_E)(\rho^{AE} \otimes \rho^{AE})\right],
$$

where $\mathbb{F}_A \in \mathcal{U}(\mathcal{H}_A \otimes \mathcal{H}_A)$ and $\mathbb{F}_E \in \mathcal{U}(\mathcal{H}_E \otimes \mathcal{H}_E)$ are flip operators exchanging the two A or E systems, respectively. Using that

$$
\operatorname{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right] = \frac{\dim(\mathcal{H}_A)}{\dim(\mathcal{H}_R)}(PU \otimes \mathbb{1}_E)\tau^{AE}(U^\dagger P \otimes \mathbb{1}_E),
$$

we find that

$$
\begin{aligned}
\int_{\mathcal{U}(\mathcal{H}_A)} &\operatorname{Tr}\left[\operatorname{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right]^2\right] d\eta(U) \\
&= \int_{\mathcal{U}(\mathcal{H}_A)} \operatorname{Tr}\left[(\mathbb{F}_A \otimes \mathbb{F}_E)\left(\operatorname{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right] \otimes \operatorname{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right]\right)\right] d\eta(U) \\
&= \frac{\dim(\mathcal{H}_A)^2}{\dim(\mathcal{H}_R)^2} \int_{\mathcal{U}(\mathcal{H}_A)} \operatorname{Tr}\left[\left(\left(\left(U^\dagger P \otimes U^\dagger P\right) \mathbb{F}_A \left(PU \otimes PU\right)\right) \otimes \mathbb{F}_E\right)\left(\tau^{AE} \otimes \tau^{AE}\right)\right] d\eta(U) \\
&= \frac{\dim(\mathcal{H}_A)^2}{\dim(\mathcal{H}_R)^2} \operatorname{Tr}\left[\left(T_{UU}\left(\mathbb{F}_R\right) \otimes \mathbb{F}_E\right)\left(\tau^{AE} \otimes \tau^{AE}\right)\right],
\end{aligned}
$$

where we used the UU-twirling channel T_{UU} from a previous lecture, and introduced the operator

$$
\mathbb{F}_R = (P \otimes P)\mathbb{F}_A(P \otimes P).
$$

It is easy to compute that

$$
\langle P_{sym}, \mathbb{F}_R \rangle_{HS} = \frac{1}{2}\left(\operatorname{Tr}\left[\mathbb{F}_R\right] + \operatorname{Tr}\left[P \otimes P\right]\right) = \frac{1}{2}\dim(\mathcal{H}_R)\left(\dim(\mathcal{H}_R) + 1\right) =: c_s,
$$

and

$$
\langle P_{asym}, \mathbb{F}_R \rangle_{HS} = \frac{1}{2}\left(\operatorname{Tr}\left[\mathbb{F}_R\right] - \operatorname{Tr}\left[P \otimes P\right]\right) = \frac{1}{2}\dim(\mathcal{H}_R)\left(1 - \dim(\mathcal{H}_R)\right) =: c_a.
$$

Using the formula for the UU-twirl from a previous lecture, we find that

$$
\begin{aligned}
T_{UU}\left(\mathbb{F}_R\right) &= c_s \frac{P_{sym}}{\operatorname{Tr}\left[P_{sym}\right]} + c_a \frac{P_{asym}}{\operatorname{Tr}\left[P_{asym}\right]} \\
&= \frac{1}{2}\left(\frac{c_s}{\operatorname{Tr}\left[P_{sym}\right]} + \frac{c_a}{\operatorname{Tr}\left[P_{asym}\right]}\right)\mathbb{1}_A \otimes \mathbb{1}_A + \frac{1}{2}\left(\frac{c_s}{\operatorname{Tr}\left[P_{sym}\right]} - \frac{c_a}{\operatorname{Tr}\left[P_{asym}\right]}\right)\mathbb{F}_A.
\end{aligned}
$$

An easy (but tedious) computation reveals that

$$\frac{1}{2}\left(\frac{c_s}{\mathrm{Tr}\left[P_{sym}\right]}+\frac{c_a}{\mathrm{Tr}\left[P_{asym}\right]}\right)=\frac{\dim(\mathcal{H}_A)}{\dim(\mathcal{H}_R)}\left(\frac{\dim(\mathcal{H}_A)-\dim(\mathcal{H}_R)}{\dim(\mathcal{H}_A)^2-1}\right)\leq\frac{1}{\dim(\mathcal{H}_R)},$$

and

$$\frac{1}{2}\left(\frac{c_s}{\mathrm{Tr}\left[P_{sym}\right]}-\frac{c_a}{\mathrm{Tr}\left[P_{asym}\right]}\right)=\frac{\dim(\mathcal{H}_A)}{\dim(\mathcal{H}_R)}\left(\frac{\dim(\mathcal{H}_R)\dim(\mathcal{H}_A)-1}{\dim(\mathcal{H}_A)^2-1}\right)\leq 1.$$

Combining the previous equations shows that

$$\begin{aligned}&\frac{\dim(\mathcal{H}_A)^2}{\dim(\mathcal{H}_R)^2}\mathrm{Tr}\left[\left(T_{UU}\left(\mathbb{F}_R\right)\otimes\mathbb{F}_E\right)\left(\tau^{AE}\otimes\tau^{AE}\right)\right]\\&\qquad\leq\mathrm{Tr}\left[\left(\mathbb{F}_A\otimes\mathbb{F}_E\right)\left(\tau^{AE}\otimes\tau^{AE}\right)\right]+\frac{1}{\dim(\mathcal{H}_R)}\mathrm{Tr}\left[\left(\mathbb{1}_A\otimes\mathbb{1}_A\otimes\mathbb{F}_E\right)\left(\tau^{AE}\otimes\tau^{AE}\right)\right]\\&\qquad=\mathrm{Tr}\left[\left(\tau^{AE}\right)^2\right]+\frac{1}{\dim(\mathcal{H}_R)}\mathrm{Tr}\left[\left(\tau^{E}\right)^2\right],\end{aligned}$$

and combining this with the computation from before we have

$$\int_{\mathcal{U}(\mathcal{H}_A)}\mathrm{Tr}\left[\mathrm{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right]^2\right]d\eta(U)\leq\mathrm{Tr}\left[\left(\tau^{AE}\right)^2\right].$$

Finally, we can use the equivalence between the $\|\cdot\|_2$-norm and the $\|\cdot\|_1$-norm together with the fact that the square root is concave to obtain

$$\begin{aligned}&\int_{\mathcal{U}(\mathcal{H}_A)}\left\|\mathrm{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right]-\frac{P}{\dim(\mathcal{H}_R)}\otimes\tau^E\right\|_1 d\eta(U)\\&\qquad\leq\int_{\mathcal{U}(\mathcal{H}_A)}\sqrt{\dim(\mathcal{H}_R)\dim(\mathcal{H}_E)\left\|\mathrm{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right]-\frac{P}{\dim(\mathcal{H}_R)}\otimes\tau^E\right\|_2^2 d\eta(U)}\\&\qquad\leq\sqrt{\dim(\mathcal{H}_R)\dim(\mathcal{H}_E)\int_{\mathcal{U}(\mathcal{H}_A)}\left\|\mathrm{Tr}_B\left[|\psi_U^{REB}\rangle\langle\psi_U^{REB}|\right]-\frac{P}{\dim(\mathcal{H}_R)}\otimes\tau^E\right\|_2^2 d\eta(U)}\\&\qquad\leq\sqrt{\dim(\mathcal{H}_R)\dim(\mathcal{H}_E)\,\mathrm{Tr}\left[\left(\tau^{AE}\right)^2\right]},\end{aligned}$$

and the proof is finished. □

3 Some technical lemmas

To prove that rates close to the coherent information are achievable for quantum communication over a quantum channel, we will need some technical lemmas. The first lemma summarizes some properties of typical projections of pure states:

Lemma 3.1. *For complex Euclidean spaces $\mathcal{H}_A, \mathcal{H}_B$ and $\mathcal{H}_E$ let $|\psi\rangle\in\mathcal{H}_A\otimes\mathcal{H}_B\otimes\mathcal{H}_E$ be some pure state and set*

$$\begin{aligned}\rho_A&=\mathrm{Tr}_{BE}\left[|\psi\rangle\langle\psi|\right]\\\rho_B&=\mathrm{Tr}_{AE}\left[|\psi\rangle\langle\psi|\right]\\\rho_E&=\mathrm{Tr}_{AB}\left[|\psi\rangle\langle\psi|\right].\end{aligned}$$

Moreover, let $\Pi_A^{n,\delta}\in\mathrm{Proj}\left(\mathcal{H}_A^{\otimes n}\right)$, $\Pi_B^{n,\delta}\in\mathrm{Proj}\left(\mathcal{H}_B^{\otimes n}\right)$ and $\Pi_E^{n,\delta}\in\mathrm{Proj}\left(\mathcal{H}_E^{\otimes n}\right)$ denote projections onto the δ-typical subspaces with respect to the marginal states $\rho_A^{\otimes n}$, $\rho_B^{\otimes n}$ and $\rho_E^{\otimes n}$, respectively, and define the (unnormalized) vector

$$|\psi_\delta^n\rangle=(\Pi_A^{n,\delta}\otimes\Pi_B^{n,\delta}\otimes\Pi_E^{n,\delta})|\psi\rangle^{\otimes n}\in H_A^{\otimes n}\otimes H_B^{\otimes n}\otimes H_E^{\otimes n}$$

Then, the following statements hold:

1. *We have*
$$\||\psi\rangle\langle\psi| - |\psi^n_\delta\rangle\langle\psi^n_\delta|\|_1 \to 0,$$
as $n \to \infty$.

2. *We have*
$$\mathrm{Tr}\left[\Pi^{n,\delta}_E\right] \leq 2^{nH(\rho_E)+n\delta}.$$

3. *We have*
$$\mathrm{Tr}\left[(\mathrm{Tr}_{A^nE^n}\left[|\psi^n_\delta\rangle\langle\psi^n_\delta|\right])^2\right] \leq 2^{-nH(\rho_B)+n\delta}$$

The proof of this lemma will use the operator-inequality

$$\begin{aligned}
&2\mathbb{1}_{\mathcal{H}_A}\otimes\mathbb{1}_{\mathcal{H}_B}\otimes\mathbb{1}_{\mathcal{H}_C} + \Pi_A\otimes\Pi_B\otimes\Pi_C \\
&\qquad \geq \Pi_A\otimes\mathbb{1}_{\mathcal{H}_B}\otimes\mathbb{1}_{\mathcal{H}_C} + \mathbb{1}_{\mathcal{H}_A}\otimes\Pi_B\otimes\mathbb{1}_{\mathcal{H}_C} + \mathbb{1}_{\mathcal{H}_A}\otimes\mathbb{1}_{\mathcal{H}_B}\otimes\Pi_C,
\end{aligned} \tag{1}$$

which holds for any triple of projections $\Pi_A \in \mathrm{Proj}\,(\mathcal{H}_A)$, $\Pi_B \in \mathrm{Proj}\,(\mathcal{H}_B)$ and $\Pi_C \in \mathrm{Proj}\,(\mathcal{H}_C)$. To prove it, note that both sides of the inequality commute and it is therefore enough to check the inequality on an orthonormal basis of joined eigenvectors. Such a basis is easily constructed from eigenbases of the individual projection operators.

Proof of Lemma 3.1. By (1) we have

$$\begin{aligned}
\langle\psi^{\otimes n}|(\Pi^{n,\delta}_A\otimes\Pi^{n,\delta}_B\otimes\Pi^{n,\delta}_E)|\psi^{\otimes n}\rangle &\geq \mathrm{Tr}\left[\Pi^{n,\delta}_A\rho_A^{\otimes n}\right] + \mathrm{Tr}\left[\Pi^{n,\delta}_B\rho_B^{\otimes n}\right] + \mathrm{Tr}\left[\Pi^{n,\delta}_E\rho_E^{\otimes n}\right] - 2 \\
&\longrightarrow 1 \text{ as } n \to \infty.
\end{aligned}$$

Then, we can use an identity from exercise sheet 6 to compute

$$\begin{aligned}
\||\psi\rangle\langle\psi|^{\otimes n} - |\psi^n_\delta\rangle\langle\psi^n_\delta|\|_1 &= \||\psi\rangle\langle\psi|^{\otimes n} - c_n\frac{|\psi^n_\delta\rangle\langle\psi^n_\delta|}{c_n}\|_1 \\
&= \sqrt{(1+c_n)^2 - 4c_n^2} \to 0,
\end{aligned}$$

as $n \to \infty$, and where we used

$$c_n = \langle\psi^n_\delta|\psi^n_\delta\rangle = \langle\psi^{\otimes n}|(\Pi^{n,\delta}_A\otimes\Pi^{n,\delta}_B\otimes\Pi^{n,\delta}_E)|\psi^{\otimes n}\rangle.$$

The second statement is a basic property of the typical projection, which we showed previously. For the third statement note that

$$\mathrm{Tr}_{A^nE^n}\left[|\psi^n_\delta\rangle\langle\psi^n_\delta|\right] \leq \Pi^{n,\delta}_B\rho_B^{\otimes n}\Pi^{n,\delta}_B \leq 2^{-nH(\rho_B)+n\delta}\Pi^{n,\delta}_B. \tag{2}$$

Here, the first inequality in (2) follows from the fact that

$$\mathrm{Tr}\left[X_B\,\mathrm{Tr}_A\left[(\Pi_A\otimes\mathbb{1}_B)Y_{AB}(\Pi_A\otimes\mathbb{1}_B)\right]\right] = \mathrm{Tr}\left[(\Pi_A\otimes X_B)Y_{AB}\right] \leq \mathrm{Tr}\left[(\mathbb{1}_A\otimes X_B)Y_{AB}\right] = \mathrm{Tr}\left[X_BY_B\right],$$

for any projection $\Pi_A \in \mathrm{Pro}(\mathcal{H}_A)$, any $X_B \in B(\mathcal{H}_B)^+$ and any $Y_{AB} \in B(\mathcal{H}_A\otimes\mathcal{H}_B)^+$. Showing that

$$\mathrm{Tr}_A\left[(\Pi_A\otimes\mathbb{1}_B)Y_{AB}(\Pi_A\otimes\mathbb{1}_B)\right] \leq Y_B.$$

The last inequality in (2) follows from a bound in lecture 9. Using again basis properties of typical projections, we find that

$$\mathrm{Tr}\left[(\mathrm{Tr}_{A^nE^n}\left[|\psi^n_\delta\rangle\langle\psi^n_\delta|\right])^2\right] \leq (2^{-nH(\rho_B)+n\delta})^2\,\mathrm{Tr}\left[\Pi^{n,\delta}_B\right] \leq 2^{-nH(\rho_B)+n\delta},$$

since for operators $X, Y \in B(\mathcal{H})$ we have

$$\mathrm{Tr}\left[(X+Y)^2\right] = \mathrm{Tr}\left[X^2\right] + 2\,\mathrm{Tr}\left[Y^{1/2}XY^{1/2}\right] + \mathrm{Tr}\left[Y^2\right] \geq \mathrm{Tr}\left[X^2\right],$$

whenever $Y \geq 0$. □

The next lemma is a simple fact about the trace distance between quantum states:

Lemma 3.2. *For every pair of quantum states $\rho, \sigma \in D(\mathcal{H})$ and any $c \in \mathbb{R}$, we have*

$$\|\rho - \sigma\|_1 \leq 2\|c\rho - \sigma\|_1.$$

Proof. We have

$$\|\rho - \sigma\|_1 \leq \|\rho - c\rho\|_1 + \|c\rho - \sigma\|_1 = |1-c| + \|c\rho - \sigma\|_1.$$

The statement of the lemma follows by realizing that

$$|1-c| = |\operatorname{Tr}[c\rho - \sigma]| \leq \|c\rho - \sigma\|_1.$$

□

Finally, we need an estimate of a certain Haar-integral, which we will prove in the exercises:

Lemma 3.3. *Consider a selfadjoint operator $H \in B(\mathcal{H}_A \otimes \mathcal{H}_B)_{sa}$ and a projection $P : \mathcal{H}_A \to \mathcal{H}_A$ with $Im(P) \subseteq \mathcal{H}_C \subset \mathcal{H}_A$. Then, we have*

$$\frac{\dim(\mathcal{H}_C)}{\operatorname{Tr}[P]} \int_{\mathcal{U}(\mathcal{H}_C)} \|(PU \otimes \mathbb{1}_{\mathcal{H}_B})H(U^\dagger P \otimes \mathbb{1}_{\mathcal{H}_B})\|_1 d\eta(U) \leq \|H\|_1.$$

Proof. Assume for each $U \in \mathcal{U}(\mathcal{H}_C)$ that $Y_U \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ satisfies $\|Y_U\|_\infty = 1$ and

$$\|(PU \otimes \mathbb{1}_{\mathcal{H}_B})H(U^\dagger P \otimes \mathbb{1}_{\mathcal{H}_B})\|_1 = \operatorname{Tr}\left[Y_U^\dagger (PU \otimes \mathbb{1}_{\mathcal{H}_B})H(U^\dagger P \otimes \mathbb{1}_{\mathcal{H}_B})\right].$$

Note that $\|Y_U\|_\infty = 1$ implies that $-\mathbb{1}_{\mathcal{H}_A} \otimes \mathbb{1}_{\mathcal{H}_B} \leq Y_U \leq \mathbb{1}_{\mathcal{H}_A} \otimes \mathbb{1}_{\mathcal{H}_B}$ implying that

$$-(U^\dagger PU \otimes \mathbb{1}_{\mathcal{H}_B}) \leq (U^\dagger P \otimes \mathbb{1}_{\mathcal{H}_B})Y_U^\dagger (PU \otimes \mathbb{1}_{\mathcal{H}_B}) \leq (U^\dagger PU \otimes \mathbb{1}_{\mathcal{H}_B}),$$

for each $U \in \mathcal{U}(\mathcal{H}_C)$. By integrating these inequalities we find that

$$-(\mathbb{1}_{\mathcal{H}_A} \otimes \mathbb{1}_{\mathcal{H}_B}) \leq \frac{\dim(\mathcal{H}_C)}{\operatorname{Tr}[P]} \int_{\mathcal{U}(\mathcal{H}_C)} (U^\dagger P \otimes \mathbb{1}_{\mathcal{H}_B})Y_U^\dagger (PU \otimes \mathbb{1}_{\mathcal{H}_B}) d\eta(U) \leq (\mathbb{1}_{\mathcal{H}_A} \otimes \mathbb{1}_{\mathcal{H}_B}).$$

This shows that

$$\|\frac{\dim(\mathcal{H}_C)}{\operatorname{Tr}[P]} \int_{\mathcal{U}(\mathcal{H}_C)} (U^\dagger P \otimes \mathbb{1}_{\mathcal{H}_B})Y_U^\dagger (PU \otimes \mathbb{1}_{\mathcal{H}_B}) d\eta(U)\|_\infty \leq 1,$$

and that

$$\begin{aligned}
&\frac{\dim(\mathcal{H}_C)}{\operatorname{Tr}[P]} \int_{\mathcal{U}(\mathcal{H}_C)} \|(PU \otimes \mathbb{1}_{\mathcal{H}_B})H(U^\dagger P \otimes \mathbb{1}_{\mathcal{H}_B})\|_1 d\eta(U) \\
&\quad = \frac{\dim(\mathcal{H}_C)}{\operatorname{Tr}[P]} \operatorname{Tr}\left[\int_{\mathcal{U}(\mathcal{H}_C)} (U^\dagger P \otimes \mathbb{1}_{\mathcal{H}_B})Y_U^\dagger (PU \otimes \mathbb{1}_{\mathcal{H}_B}) d\eta(U) \cdot H\right] \leq \|H\|_1.
\end{aligned}$$

□

4 Achieving rates close to the coherent information

Now, we can proceed with the main theorem in this lecture:

Theorem 4.1. *Let $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ denote a quantum channel and $\sigma \in D(\mathcal{H}_A)$ a quantum state. Then, any rate*

$$0 \leq R < I_c(\sigma; T),$$

is achievable for entanglement generation over the quantum channel T.

Proof. Throughout the proof, we will fix a $\delta > 0$, and we will show that any rate

$$0 \leq R < I_c(\sigma; T) - 3\delta,$$

is achievable for entanglement generation over the quantum channel T. Let $V : \mathcal{H}_A \to \mathcal{H}_E \otimes \mathcal{H}_B$ denote a Stinespring isometry of T and let $|\phi^{AA'}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_A$ denote the purification of $\sigma \in D(\mathcal{H}_A)$ given by

$$|\phi^{AA'}\rangle = \text{vec}\left(\sqrt{\sigma}\right).$$

Next, consider the reduced quantum states

$$\begin{aligned}
\tau^A &= \text{Tr}_{EB}\left[(\mathbb{1}_A \otimes V)\,|\phi^{AA'}\rangle\langle\phi^{AA'}|\left(\mathbb{1}_A \otimes V^\dagger\right)\right] = \text{Tr}_{A'}\left[|\phi^{AA'}\rangle\langle\phi^{AA'}|\right] \\
\tau^E &= \text{Tr}_{AB}\left[(\mathbb{1}_A \otimes V)\,|\phi^{AA'}\rangle\langle\phi^{AA'}|\left(\mathbb{1}_A \otimes V^\dagger\right)\right] = T^c(\sigma) \\
\tau^B &= \text{Tr}_{AE}\left[(\mathbb{1}_A \otimes V)\,|\phi^{AA'}\rangle\langle\phi^{AA'}|\left(\mathbb{1}_A \otimes V^\dagger\right)\right] = T(\sigma),
\end{aligned}$$

of the pure quantum state

$$|\tau^{AEB}\rangle = (\mathbb{1}_A \otimes V)|\phi^{AA'}\rangle.$$

For brevity, let us set

$$|\tau^n\rangle = \left(|\tau^{AEB}\rangle\right)^{\otimes n} \in H_A^{\otimes n} \otimes H_E^{\otimes n} \otimes H_B^{\otimes n}.$$

For every $n \in \mathbb{N}$ we consider the projections $\Pi_A^{n,\delta} \in \text{Proj}\left(\mathcal{H}_A^{\otimes n}\right)$, $\Pi_B^{n,\delta} \in \text{Proj}\left(\mathcal{H}_B^{\otimes n}\right)$ and $\Pi_E^{n,\delta} \in \text{Proj}\left(\mathcal{H}_E^{\otimes n}\right)$ onto the δ-typical subspaces $\mathcal{H}_A^{n,\delta} \subset \mathcal{H}_A^{\otimes n}$, $\mathcal{H}_B^{n,\delta} \subset \mathcal{H}_B^{\otimes n}$ and $\mathcal{H}_E^{n,\delta} \subset \mathcal{H}_E^{\otimes n}$ with respect to the marginal states $\tau_A^{\otimes n}$, $\tau_B^{\otimes n}$ and $\tau_E^{\otimes n}$, respectively. Then, we define the (unnormalized) vectors

$$|\tau_\delta^n\rangle = (\Pi_A^{n,\delta} \otimes \Pi_E^{n,\delta} \otimes \Pi_B^{n,\delta})|\tau_n\rangle \in \mathcal{H}_A^{n,\delta} \otimes \mathcal{H}_E^{n,\delta} \otimes \mathcal{H}_B^{n,\delta},$$

and we denote its marginal by

$$\tau_{n,\delta}^E = \text{Tr}_{A^n B^n}\left[|\tau_\delta^n\rangle\langle\tau_\delta^n|\right].$$

Moreover, for any unitary $U \in \mathcal{U}\left(\mathcal{H}_A^{n,\delta}\right)$ we define the (unnormalized) vectors

$$|\psi_U^n\rangle = \sqrt{\frac{\dim(H_A^{n,\delta})}{\dim(\mathcal{H}_R^n)}}\left(PU \otimes \mathbb{1}_{\mathcal{H}_E^{\otimes n}} \otimes \mathbb{1}_{\mathcal{H}_B^{\otimes n}}\right)|\tau^n\rangle \in \mathcal{H}_R^{\otimes n} \otimes \mathcal{H}_E^{\otimes n} \otimes \mathcal{H}_B^{\otimes n},$$

and

$$|\psi_U^{n,\delta}\rangle = \sqrt{\frac{\dim(H_A^{n,\delta})}{\dim(\mathcal{H}_R^n)}}\left(PU \otimes \mathbb{1}_{\mathcal{H}_E^{n,\delta}} \otimes \mathbb{1}_{\mathcal{H}_B^{n,\delta}}\right)|\tau_\delta^n\rangle \in \mathcal{H}_R^{\otimes n} \otimes \mathcal{H}_E^{n,\delta} \otimes \mathcal{H}_B^{n,\delta},$$

for some fixed projection $P : \mathcal{H}_A^{\otimes n} \to \mathcal{H}_A^{\otimes n}$ with $\mathcal{H}_R^n = \text{Im}(P) \subset \mathcal{H}_A^{n,\delta}$ being some subspace of dimension $\dim(\mathcal{H}_R^n) = 2^{nR}$. Finally, we define the pure quantum state

$$|\tilde{\psi}_U^n\rangle = \frac{1}{\sqrt{\langle\psi_U^n|\psi_U^n\rangle}}|\psi_U^n\rangle \in \mathcal{H}_R^{\otimes n} \otimes \mathcal{H}_E^{\otimes n} \otimes \mathcal{H}_B^{\otimes n},$$

for every $n \in \mathbb{N}$ and each $U \in \mathcal{U}\left(\mathcal{H}_A^{n,\delta}\right)$.

Note that for every $n \in \mathbb{N}$ and $U \in \mathcal{U}\left(\mathcal{H}_A^{n,\delta}\right)$ we have

$$|\tilde{\psi}_U^n\rangle = \left(\mathbb{1}_{R^n} \otimes V^{\otimes n}\right)\left(|\phi_{RA'}^{n,U}\rangle\langle\phi_{RA'}^{n,U}|\right),$$

for the pure quantum states $|\phi_{RA'}^{n,U}\rangle \in \mathcal{H}_R^n \otimes \mathcal{H}_A^{\otimes n}$ arising from normalizing the vectors

$$\sqrt{\frac{\dim(H_A^{n,\delta})}{\dim(\mathcal{H}_R^n)}}\left(PU \otimes \mathbb{1}_{\mathcal{H}_A^{\otimes n}}\right)|\phi^{AA'}\rangle^{\otimes n}.$$

We will now show that for each $n \in \mathbb{N}$, there exists a unitary $U_n \in \mathcal{U}\left(\mathcal{H}_A^{n,\delta}\right)$ such that

$$\| \operatorname{Tr}_{B^n}\left[|\tilde{\psi}_{U_n}^n\rangle\langle\tilde{\psi}_{U_n}^n|\right] - \frac{P}{2^{nR}} \otimes (\tau^E)^{\otimes n}\|_1 \to 0,$$

as $n \to \infty$. Using Theorem 2.1 and identifying $\mathcal{H}_R^n$ with $(\mathbb{C}^2)^{\otimes Rn}$, we find a sequence of quantum channels $D_n : B(\mathcal{H}_B^{\otimes n}) \to B(\mathcal{H}_R^n)$ satisfying

$$F(\omega_2^{\otimes Rn}, (\mathrm{id}_2^{\otimes Rn} \otimes D_n \circ T^{\otimes n})\left(|\phi_{RA'}^{n,U}\rangle\langle\phi_{RA'}^{n,U}|\right)) \to 1,$$

as $n \to \infty$. We conclude that the rate R is achievable for entanglement generation using T.

To finish the proof, we first apply Lemma 3.2 to estimate

$$\| \operatorname{Tr}_{B^n}\left[|\tilde{\psi}_U^n\rangle\langle\tilde{\psi}_U^n|\right] - \frac{P}{2^{nR}} \otimes (\tau^E)^{\otimes n}\|_1 \leq 2\| \operatorname{Tr}_{B^n}\left[|\psi_U^n\rangle\langle\psi_U^n|\right] - \frac{P}{2^{nR}} \otimes (\tau^E)^{\otimes n}\|_1,$$

for every $n \in \mathbb{N}$ and each $U \in \mathcal{U}\left(H_A^{n,\delta}\right)$. To estimate the right-hand side in the previous inequality, we use the triangle inequality to obtain

$$\begin{aligned}\| \operatorname{Tr}_{B^n}\left[|\psi_U^n\rangle\langle\psi_U^n|\right] - \frac{P}{2^{nR}} \otimes (\tau^E)^{\otimes n}\|_1 &\leq \| \operatorname{Tr}_{B^n}\left[|\psi_U^n\rangle\langle\psi_U^n|\right] - \operatorname{Tr}_{B^n}\left[|\psi_U^{n,\delta}\rangle\langle\psi_U^{n,\delta}|\right]\|_1 \\ &+ \| \operatorname{Tr}_{B^n}\left[|\psi_U^{n,\delta}\rangle\langle\psi_U^{n,\delta}|\right] - \frac{P}{2^{nR}} \otimes \tau_{n,\delta}^E\|_1 \\ &+ \|\frac{P}{2^{nR}} \otimes \tau_{n,\delta}^E - \frac{P}{2^{nR}} \otimes (\tau^E)^{\otimes n}\|_1.\end{aligned}$$

We will now derive upper bounds on the Haar-integrals of the three summands in the last inequality. First, note that by monotonicity of the trace-distance under partial traces, Lemma 3.2 and 3.1 we have

$$\begin{aligned}\|\frac{P}{2^{nR}} \otimes \tau_{n,\delta}^E - \frac{P}{2^{nR}} \otimes (\tau^E)^{\otimes n}\|_1 &\leq \|\tau_{n,\delta}^E - (\tau^E)^{\otimes n}\|_1 \\ &\leq \||\tau_\delta^n\rangle\langle\tau_\delta^n| - |\tau^n\rangle\langle\tau^n|\|_1 \to 0,\end{aligned}$$

as $n \to \infty$. Furthermore, by Lemma 3.3 we have that

$$\begin{aligned}&\int_{\mathcal{U}\left(\mathcal{H}_A^{n,\delta}\right)} \| \operatorname{Tr}_{B^n}\left[|\psi_U^n\rangle\langle\psi_U^n|\right] - \operatorname{Tr}_{B^n}\left[|\psi_U^{n,\delta}\rangle\langle\psi_U^{n,\delta}|\right]\|_1 d\eta(U) \\ &\leq \| \operatorname{Tr}_{B^n}\left[|\tau^n\rangle\langle\tau^n|\right] - \operatorname{Tr}_{B^n}\left[|\tau_\delta^n\rangle\langle\tau_\delta^n|\right]\|_1 \to 0,\end{aligned}$$

as $n \to \infty$. Finally, we note that by Theorem 2.2 we have

$$\begin{aligned}&\int_{\mathcal{U}\left(\mathcal{H}_A^{n,\delta}\right)} \| \operatorname{Tr}_{B^n}\left[|\psi_U^{n,\delta}\rangle\langle\psi_U^{n,\delta}|\right] - \frac{P}{2^{nR}} \otimes \tau_{n,\delta}^E\|_1 d\eta(U) \\ &\leq \sqrt{2^{nR}\dim(\mathcal{H}_E^{n,\delta})\operatorname{Tr}\left[\left(\tau_{n,\delta}^B\right)^2\right]} \\ &\leq \sqrt{2^{n(R-H(T(\sigma))+H(T^c(\sigma))+3\delta)}} \to 0,\end{aligned}$$

as $n \to \infty$ under the assumptions on the rate R from above. We conclude that

$$\int_{\mathcal{U}\left(\mathcal{H}_A^{n,\delta}\right)} \| \operatorname{Tr}_{B^n}\left[|\psi_U^n\rangle\langle\psi_U^n|\right] - \frac{P}{2^{nR}} \otimes (\tau^E)^{\otimes n}\|_1 d\eta(U) \to 0,$$

and hence there exists a sequence $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}\left(\mathcal{H}_A^{n,\delta}\right)^{\mathbb{N}}$ satisfying

$$\| \operatorname{Tr}_{B^n}\left[|\psi_{U_n}^n\rangle\langle\psi_{U_n}^n|\right] - \frac{P}{2^{nR}} \otimes (\tau^E)^{\otimes n}\|_1 \to 0,$$

as $n \to \infty$. Finally, we can combine this with the estimates from above to see that

$$\| \operatorname{Tr}_{B^n}\left[|\tilde{\psi}_{U_n}^n\rangle\langle\tilde{\psi}_{U_n}^n|\right] - \frac{P}{2^{nR}} \otimes (\tau^E)^{\otimes n}\|_1 \to 0,$$

as $n \to \infty$. By the argument from before, this finishes the proof. □

In the previous lecture, we have seen that the entanglement generation capacity coincides with the quantum capacity, and together with the previous theorem we conclude that

$$I_c\left(\sigma; T\right) \leq Q(T),$$

for any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and any quantum state $\sigma \in D(\mathcal{H}_A)$. Optimizing over $\sigma \in D(\mathcal{H}_A)$ and applying the resulting bound for the quantum channel $T^{\otimes k}$ instead of T implies[1] the following corollary:

Corollary 4.2. *For any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ we have*

$$\limsup_{k\to\infty} \frac{1}{k} I_c\left(T^{\otimes k}\right) \leq Q(T).$$

[1] by using that $Q(T^{\otimes k}) = kQ(T)$.