Quantum information theory (MAT4430) **Spring 2021**

Lecture 18: LSD theorem (part 2), superactivation and classical assistance

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In the final lecture, we will prove the converse part of the LSD theorem. We will then study a striking consequence of this theorem: Superactivation. Although two channels each have zero quantum capacity their tensor product can have non-zero quantum capacity. One way of putting this is that allowing for a side-channel with zero capacity to be used by the sender and the receiver can increase the quantum capacity. Along similar lines, we will then look at classical side-channels either from the sender to the receiver or from the receiver to the sender. We will see that in the first case the classical side-channel does not increase the quantum capacity, but in the second case it does.

1 The proof of the LSD theorem

We will need the following theorem, which we have shown in the exercises:

Theorem 1.1 (Data-processing inequality for the coherent information). *For quantum channels $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ and $S : B(\mathcal{H}_2) \to B(\mathcal{H}_3)$ we have*

$$I_c(\sigma, S \circ T) \leq I_c(\sigma, T),$$

for any $\sigma \in D(\mathcal{H}_1)$.

Proof. Exercise. □

Now, we can show the main result of this lecture:

Theorem 1.2 (Lloyd, Shor, Devetak). *For any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ we have*

$$Q(T) = \lim_{k\to\infty} \frac{1}{k} I_c\left(T^{\otimes k}\right).$$

Proof. In the previous lecture, we have seen that

$$\limsup_{k\to\infty} \frac{1}{k} I_c\left(T^{\otimes k}\right) \leq Q_{EG}(T) = Q(T),$$

for any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$. To prove the remaining inequality assume that $R > 0$ is an achievable rate for entanglement generation via T. Therefore, we have for every $n \in \mathbb{N}$ there is an (n, m_n, δ_n)-coding scheme for entanglement generation consisting of a quantum channel

$$D_n : B(\mathcal{H}_B^{\otimes n}) \to B\left((\mathbb{C}^2)^{\otimes m_n}\right),$$

together with a pure quantum state

$$|\phi_n\rangle \in (\mathbb{C}^2)^{\otimes m_n} \otimes (\mathcal{H}_A)^{\otimes n},$$

such that $R = \lim_{n\to\infty} m_n/n$ and

$$F\left(\omega_2^{\otimes m_n}, \left(\text{id}_2^{\otimes m_n} \otimes D_n \circ T^{\otimes n}\right)(|\phi_n\rangle\langle\phi_n|)\right) \geq 1 - \delta_n \to 1,$$

as $n \to \infty$. By the Fuchs-van-de-Graaf inequalities we find that

$$\|\omega_2^{\otimes m_n} - \left(\text{id}_2^{\otimes m_n} \otimes D_n \circ T^{\otimes n}\right)(|\phi_n\rangle\langle\phi_n|)\|_1 = \epsilon_n \to 0,$$

as $n \to \infty$, and (by taking the partial trace) we also have

$$\| \frac{\mathbb{1}_2^{\otimes m_n}}{2^{m_n}} - \left(D_n \circ T^{\otimes n}\right)(\rho_n) \|_1 \leq \epsilon_n,$$

where we set $\rho_n \in (\mathcal{H}_A)^{\otimes n}$ to denote the partial traces of $|\phi_n\rangle\langle\phi_n|$ over the reference system. Using Fannes' inequality (which we proved in exercise 4 on sheet 8), we find that

$$H\left(\left(\text{id}_2^{\otimes m_n} \otimes D_n \circ T^{\otimes n}\right)(|\phi_n\rangle\langle\phi_n|)\right) \leq 2m_n\epsilon_n + 1$$

and that

$$m_n - H\left(\left(D_n \circ T^{\otimes n}\right)(\rho_n)\right) \leq m_n\epsilon_n + 1,$$

whenever $n \in \mathbb{N}$ is chosen large enough (to guarantee that $\eta(\epsilon_n) \leq 1$). Since both $|\phi_n\rangle$ and $\text{vec}\left(\sqrt{\rho_n}\right)$ are purifications of the same quantum state ρ_n, we can use the Schmidt-decomposition to show that

$$\left(\text{id}_2^{\otimes m_n} \otimes D_n \circ T^{\otimes n}\right)(|\phi_n\rangle\langle\phi_n|),$$

and

$$\left(\text{id}_A^{\otimes n} \otimes D_n \circ T^{\otimes n}\right)\left(\text{vec}\left(\sqrt{\rho_n}\right)\text{vec}\left(\sqrt{\rho_n}\right)^\dagger\right)$$

have the same non-zero spectrum. Therefore, we have

$$\begin{aligned} &H\left(\left(\text{id}_2^{\otimes m_n} \otimes D_n \circ T^{\otimes n}\right)(|\phi_n\rangle\langle\phi_n|)\right) \\ &\quad = H\left(\left(\text{id}_A^{\otimes n} \otimes D_n \circ T^{\otimes n}\right)\left(\text{vec}\left(\sqrt{\rho_n}\right)\text{vec}\left(\sqrt{\rho_n}\right)^\dagger\right)\right). \end{aligned}$$

Using the data processing inequality of the coherent information (see exercises), we find that

$$\begin{aligned} I_c(T^{\otimes n}) &\geq I_c(D_n \circ T^{\otimes n}) \\ &\geq I_c(\rho_n, D \circ T^{\otimes n}) \\ &= H\left(\left(D_n \circ T^{\otimes n}\right)(\rho_n)\right) - H\left(\left(\text{id}_A^{\otimes n} \otimes D_n \circ T^{\otimes n}\right)\left(\text{vec}\left(\sqrt{\rho_n}\right)\text{vec}\left(\sqrt{\rho_n}\right)^\dagger\right)\right) \\ &= H\left(\left(D_n \circ T^{\otimes n}\right)(\rho_n)\right) - H\left(\left(\text{id}_2^{\otimes m_n} \otimes D_n \circ T^{\otimes n}\right)(|\phi_n\rangle\langle\phi_n|)\right) \\ &\geq m_n(1 - 3\delta_n) - 2. \end{aligned}$$

We conclude that

$$\liminf_{n\to\infty} \frac{1}{n} I_c(T^{\otimes n}) \geq \liminf_{n\to\infty} \left(\frac{m_n}{n}(1 - 3\delta_n) - \frac{2}{n}\right) = R.$$

We conclude that

$$\limsup_{k\to\infty} \frac{1}{k} I_c\left(T^{\otimes k}\right) \leq Q(T) \leq \liminf_{k\to\infty} \frac{1}{k} I_c(T^{\otimes k}),$$

and hence we have shown that

$$Q(T) = \lim_{k\to\infty} \frac{1}{k} I_c\left(T^{\otimes k}\right).$$

□

The LSD-theorem derives a formula for the quantum capacity of a quantum channel, but again this formula is not satisfactory as it involves a regularization. It is currently not known whether there exists a different type of formula not involving any regularization, and it is not even known whether the quantum capacity is a computable quantity in the sense of Turing.

In the following, we collect some examples of quantum channels $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ for which the capacity is known:

- If $\vartheta_B \circ T$ is completely positive, then $Q(T) = 0$.
- If T is antidegradable, then $Q(T) = 0$.
- If T is degradable, then $Q(T) = I_c(T)$.
- In particular, the erasure channel $E_\lambda : B(\mathbb{C}^d) \to B(\mathbb{C}^{d+1})$ for $\lambda \in [0,1]$ given by

  $$E_\lambda(X) = (1-\lambda)X \oplus 0 + \lambda \operatorname{Tr}[X] \, |d+1\rangle\langle d+1|.$$

  satisfies

  $$Q(E_\lambda) = (1-2\lambda)\log(d).$$

- If $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ and the quantum channel T is Pauli-diagonal, i.e., we have

  $$T = p_0 \mathrm{id}_2 + p_1 \operatorname{Ad}_{\sigma_X} + p_2 \operatorname{Ad}_{\sigma_Y} + p_3 \operatorname{Ad}_{\sigma_Z},$$

  for $p \in \mathcal{P}(\{0,1,2,3\})$, then

  $$Q(T) \geq 1 - H(p).$$

  This inequality is called the *Hashing bound.*

2 Superactivation

The following theorem is due to Graeme Smith and John Yard.

Theorem 2.1. *Consider an isometry*

$$V : \mathbb{C}^{d_A} \to \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_E},$$

and the complementary quantum channels $T : B(\mathbb{C}^{d_A}) \to B(\mathbb{C}^{d_B})$ and $T^c : B(\mathbb{C}^{d_A}) \to B(\mathbb{C}^{d_E})$ given by

$$T(X) = \operatorname{Tr}_E\left[VXV^\dagger\right] \quad \text{and} \quad T^c(X) = \operatorname{Tr}_B\left[VXV^\dagger\right].$$

Moreover, let $\{p_i, \rho_i\}_{i=1}^N$ denote an ensemble of quantum states with a probability distribution $p \in \mathcal{P}(\{1,\ldots,N\})$ and $\rho_i \in D(\mathbb{C}^{d_A})$ for every $i \in \{1,\ldots,N\}$ and let $D = Nd_A$. Then, we have

$$I_c\left(T \otimes E_{\frac{1}{2}}\right) \geq \frac{1}{2}\chi\left(\{p_i, T(\rho_i)\}\right) - \frac{1}{2}\chi\left(\{p_i, T^c(\rho_i)\}\right),$$

where $E_{\frac{1}{2}} : B(\mathbb{C}^D) \to B(\mathbb{C}^{D+1})$ is the erasure channel defined by

$$E_{\frac{1}{2}}(X) = \frac{1}{2}X \oplus 0 + \frac{1}{2}|D+1\rangle\langle D+1|.$$

Proof. Consider the classical-quantum state

$$\rho_{CA} = \sum_{i=1}^N p_i |i\rangle\langle i| \otimes \rho_i^A \in D(\mathbb{C}^N \otimes \mathbb{C}^{d_A})$$

and its purification

$$|\psi_{C'CA'A}\rangle = \sum_{i=1}^N \sqrt{p_i}|i\rangle \otimes |i\rangle \otimes |\phi_i^{A'A}\rangle \in \mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_A},$$

where each $|\phi_i^{A'A}\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_A}$ denotes a purification of $\rho_i^A \in D(\mathbb{C}^{d_A})$. Next, we define the pure quantum state

$$|\tau_{C'CA'EB}\rangle = (\mathbb{1}_{C'} \otimes \mathbb{1}_C \otimes \mathbb{1}_{A'} \otimes V)\left(|\psi_{C'CA'A}\rangle\right).$$

It is easy to verify that

$$\tau_{CE} = \mathrm{Tr}_{C'A'B}\left[|\tau_{C'CA'EB}\rangle\langle\tau_{C'CA'EB}|\right] = \sum_{i=1}^{N} p_i |i\rangle\langle i| \otimes T^c\left(\rho_i^A\right)$$

$$\tau_{CB} = \mathrm{Tr}_{C'A'E}\left[|\tau_{C'CA'EB}\rangle\langle\tau_{C'CA'EB}|\right] = \sum_{i=1}^{N} p_i |i\rangle\langle i| \otimes T\left(\rho_i^A\right)$$

$$\tau_{E} = \mathrm{Tr}_{C'CA'B}\left[|\tau_{C'CA'EB}\rangle\langle\tau_{C'CA'EB}|\right] = \sum_{i=1}^{N} p_i T^c\left(\rho_i^A\right)$$

$$\tau_{B} = \mathrm{Tr}_{C'CA'E}\left[|\tau_{C'CA'EB}\rangle\langle\tau_{C'CA'EB}|\right] = \sum_{i=1}^{N} p_i T\left(\rho_i^A\right),$$

and by the definition of the Holevo quantity (and properties of the von Neumann entropy) we have

$$\chi\left(\{p_i, T(\rho_i)\}\right) = H(\tau_B) - H(\tau_{CB}) + H(p),$$

and

$$\chi\left(\{p_i, T^c(\rho_i)\}\right) = H(\tau_E) - H(\tau_{CE}) + H(p).$$

Finally, we consider the quantum state

$$\rho_{C'A'A} = \mathrm{Tr}_C\left[|\psi_{C'CA'A}\rangle\langle\psi_{C'CA'A}|\right],$$

and, by identifying $\mathbb{C}^D = \mathbb{C}^N \otimes \mathbb{C}^{d_A}$, we can compute

$$\begin{aligned}
I_c(E_{\frac{1}{2}} \otimes T) &\geq I_c(\rho_{C'A'A}, E_{\frac{1}{2}} \otimes T) \\
&= H\left(\left(E_{\frac{1}{2}} \otimes T\right)(\rho_{C'A'A})\right) - H\left(\left(E_{\frac{1}{2}} \otimes T^c\right)(\rho_{C'A'A})\right) \\
&= \frac{1}{2} H\left((\mathrm{id}_{C'A'} \otimes T)(\rho_{C'A'A})\right) + \frac{1}{2} H\left(T(\rho_A)\right) \\
&\quad - \frac{1}{2} H\left((\mathrm{id}_{C'A'} \otimes T^c)(\rho_{C'A'A})\right) - \frac{1}{2} H\left(T^c(\rho_A)\right) \\
&= \frac{1}{2} H(\tau_{C'A'B}) + \frac{1}{2} H(\tau_B) - \frac{1}{2} H(\tau_{C'A'E}) - \frac{1}{2} H(\tau_E) \\
&= \frac{1}{2} H(\tau_{CE}) + \frac{1}{2} H(\tau_B) - \frac{1}{2} H(\tau_{CB}) - \frac{1}{2} H(\tau_E) \\
&= \frac{1}{2} \chi\left(\{p_i, T(\rho_i)\}\right) - \frac{1}{2} \chi\left(\{p_i, T^c(\rho_i)\}\right)
\end{aligned}$$

□

To find an explicit example for superactivation, we could use a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ such that $\vartheta_B \circ T$ is completely positive and such that

$$\frac{1}{2} \chi\left(\{p_i, T(\rho_i)\}\right) - \frac{1}{2} \chi\left(\{p_i, T^c(\rho_i)\}\right) > 0,$$

for some ensemble $\{p_i, \rho_i\}_{i=1}^N$ with $\rho_i \in D(\mathcal{H}_A)$. Indeed such quantum channels exist, and you may look into the book by Watrous to see an explicit example of a quantum channel

$T : B(\mathbb{C}^4) \rightarrow B(\mathbb{C}^4)$ satisfying these properties. From the transposition bound we know that $Q(T) = 0$, but the previous theorem implies that

$$Q(T \otimes E_{\frac{1}{2}}) \geq I_c\left(T \otimes E_{\frac{1}{2}}\right) > 0,$$

for some erasure channel $E_{\frac{1}{2}} : B(\mathbb{C}^D) \rightarrow B(\mathbb{C}^{D+1})$ with erasure probability $1/2$. We have seen earlier that $E_{\frac{1}{2}}$ is antidegradable and hence we have $Q(E_{\frac{1}{2}}) = 0$. From this we can conclude that surprising fact that the tensor product $T_1 \otimes T_2$ of two quantum channels can have strictly positive quantum capacity although both T_1 and T_2 have zero quantum capacity. This phenomenon is called *superactivation*!

By a similar argument (see the book by Watrous for the details) to what we did for the classical capacity, the following corollary can be obtained from an example of superactivation:

Corollary 2.2. *There is a quantum channel $T : B(\mathcal{H}_A) \rightarrow B(\mathcal{H}_B)$ satisfying*

$$I_c(T \otimes T) > 2I_c(T).$$

The coherent information is not *additive!*

3 Does classical communication help?

In most quantum communication scenarios happening on earth, it is a reasonable assumption that the sender and receiver can communicate classically. How does this affect the achivable rates for quantum communication? We will consider to basic scenarios of such an assistance: Forward classical communication from the sender to the receiver and backward classical communication from the receiver to the sender. We will see that while the former does not increase quantum communication rates, the latter does.

3.1 Classical forward communication does not help!

How should we model forward classical communication from the sender to the receiver in the quantum communcation scenario? Well, instead of using an encoding quantum channel the sender could use a general encoding instrument and the decoding channel that the receiver applies could depend on the classical information in the output of the instrument. Indeed, this captures the most general scenario of classical forward communication. The next definition makes this precise:

Definition 3.1 (Coding schemes assisted by classical forward communication)**.** *Let $T : B(\mathcal{H}_A) \rightarrow B(\mathcal{H}_B)$ denote a quantum channel. An (n, m, δ)-coding scheme for* quantum communication over T assisted by forward classical communication *is given by an instrument $\{E_i\}_{i=1}^K$ of completely positive maps*

$$E_i : B\left((\mathbb{C}^2)^{\otimes m}\right) \rightarrow B(\mathcal{H}_A^{\otimes n})$$

such that $\sum_{i=1}^K E_i$ is a quantum channel, and quantum channels

$$D_i : B(\mathcal{H}_B^{\otimes n}) \rightarrow B\left((\mathbb{C}^2)^{\otimes m}\right),$$

for each $i \in \{1, \ldots, K\}$ such that

$$\|\text{id}_2^{\otimes m} - \sum_{i=1}^K D_i \circ T^{\otimes n} \circ E_i\|_\diamond \leq \delta.$$

As always, we define a capacity as follows:

Definition 3.2 (Quantum capacity assisted by classical forward communication)**.** *We call a rate* $R \geq 0$ *achievable for* quantum communication assisted by classical forward communication *over the quantum channel* $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ *if for every* $n \in \mathbb{N}$*, there exists an* (n, m_n, δ_n)*-coding scheme such that*

$$R = \lim_{n\to\infty} \frac{m_n}{n} \quad \textit{and} \quad \lim_{n\to\infty} \delta_n = 0.$$

The quantum capacity assisted by classical forward communication *of a quantum channel* $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ *is given by*

$$Q_{\to}(T) = \sup\{R \geq 0 \ : \ R \textit{ achievable rate for quantum comm. assisted by class. forward comm.}\}.$$

We will need the following lemma about the fidelity:

Lemma 3.3. *Let* $T : B(\mathcal{H}_A) \to B(\mathcal{H}_R)$ *denote a quantum channel and* $E : B(\mathcal{H}_R) \to B(\mathcal{H}_A)$ *a completely positive map satisfying* $\text{Tr}\left[E(\mathbb{1}_R)\right] = d_R$ *and*

$$F\left(\omega_R, \left(\text{id}_R \otimes T \circ E\right)(\omega_R)\right)^2 \geq 1 - \epsilon,$$

for some $\epsilon \in (0, 1]$*. Then, there exists a pure quantum state* $|\psi_{RA}\rangle \in \mathcal{H}_R \otimes \mathcal{H}_A$ *such that*

$$F\left(\omega_R, \left(\text{id}_R \otimes T\right)(|\psi_{RA}\rangle\langle\psi_{RA}|)\right)^2 \geq 1 - 2\epsilon.$$

Proof. Consider Kraus decompositions

$$T = \sum_{i=1}^{N} \text{Ad}_{K_i} \quad \text{and} \quad E = \sum_{j=1}^{N} \text{Ad}_{L_j},$$

with Kraus operators (which might be zero)

$$K_i : \mathcal{H}_A \to \mathcal{H}_R \quad \text{and} \quad L_i : \mathcal{H}_R \to \mathcal{H}_A,$$

for every $i, j \in \{1, \ldots, N\}$ such that

$$\text{Tr}\left[K_i L_j\right] = 0,$$

whenever $i \neq j$. The existence of such Kraus decompositions follows by considering general sets of Kraus operators $\{\tilde{K}_i\}_{i=1}^N$ and $\{\tilde{L}_j\}_{j=1}^M$ and the matrix $X \in B(\mathbb{C}^N, \mathbb{C}^M)$ with entries

$$X_{ij} = \text{Tr}\left[K_i L_j\right].$$

By the singular value decomposition, there exist unitaries $U \in \mathcal{U}\left(\mathbb{C}^N\right)$ and $U' \in \mathcal{U}\left(\mathbb{C}^M\right)$ such that

$$\sum_{kl} U_{ik}\overline{U'_{lj}}\, \text{Tr}\left[\tilde{K}_k \tilde{L}_l\right] = 0,$$

whenever $i \neq j$. Now, the operators $K_i = \sum_{k=1}^N U_{ik}\tilde{K}_k$ and $L_j = \sum_{l=1}^M \overline{U'_{lj}}\tilde{L}_l$ define new sets of Kraus operators with the desired property.

By a lemma from lecture 9 we have

$$F\left(\omega_R, \left(\text{id}_R \otimes T \circ E\right)(\omega_R)\right)^2 = \frac{1}{d_R^2}\sum_{i=1}^{N} |\,\text{Tr}\left[K_i L_i\right]|^2,$$

and in the following we may assume that K_i and L_i are non-zero (otherwise restrict the sum to only include the non-zero terms). Then, we have

$$F\left(\omega_R, \left(\text{id}_R \otimes T \circ E\right)(\omega_R)\right)^2 = \frac{1}{d_R^2}\sum_{i=1}^{N} p_i \frac{|\,\text{Tr}\left[K_i L_i\right]|^2}{p_i} \geq 1 - \epsilon,$$

where we introduced

$$p_i = \frac{1}{d_R} \operatorname{Tr}\left[L_i L_i^\dagger\right] > 0.$$

Since $\sum_i p_i = 1$, there exists some $i \in \{1, \ldots, N\}$ such that

$$\frac{|\operatorname{Tr}[K_i L_i]|^2}{d_R^2 p_i} = \frac{|\operatorname{Tr}[K_i L_i]|^2}{d_R \operatorname{Tr}\left[L_i L_i^\dagger\right]} \geq 1 - \epsilon.$$

Now, we define the operator

$$L = \frac{\sqrt{d_R} L_i}{\sqrt{\operatorname{Tr}\left[L_i L_i^\dagger\right]}},$$

and the pure quantum state

$$|\psi\rangle = (\mathbb{1}_R \otimes L)|\Omega_R\rangle,$$

where $|\Omega_R\rangle$ denotes the normalized maximally entangled state. It is then easy to verify that

$$F\left(\omega_R, (\mathrm{id}_R \otimes T)\left(|\psi_{RA}\rangle\langle\psi_{RA}|\right)\right)^2 \geq \frac{1}{d_R^2} |\operatorname{Tr}[K_i L]|^2 = \frac{|\operatorname{Tr}[K_i L_i]|^2}{d_R \operatorname{Tr}\left[L_i L_i^\dagger\right]} \geq 1 - \epsilon.$$

□

Now, we show that forward classical communication does not help:

Theorem 3.4. *For any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ we have*

$$Q_{\to}(T) = Q(T).$$

Proof. Clearly, we have $Q(T) \leq Q_{\to}(T)$ since the parties can just choose not to communicate. To see the other inequality assume that $R \geq 0$ is an achievable rate for quantum communication assisted by forward classical communication. Consider a sequence of (n, m_n, δ_n)-coding schemes for quantum communication assisted by forward classical communication for each $n \in \mathbb{N}$ such that

$$R = \lim_{n\to\infty} \frac{m_n}{n},$$

given by a sequence of encoding instruments $\{E_i^{(n)}\}_{i=1}^{K_n}$ with

$$E_i^{(n)} : B\left((\mathbb{C}^2)^{\otimes m_n}\right) \to B(\mathcal{H}_A^{\otimes n})$$

such that $\sum_{i=1}^{K_n} E_i^{(n)}$ is a quantum channel for each $n \in \mathbb{N}$, and decoding quantum channels

$$D_i^{(n)} : B(\mathcal{H}_B^{\otimes n}) \to B\left((\mathbb{C}^2)^{\otimes m_n}\right),$$

for each $i \in \{1, \ldots, K_n\}$ such that

$$\|\mathrm{id}_2^{\otimes m_n} - \sum_{i=1}^{K_n} D_i^{(n)} \circ T^{\otimes n} \circ E_i^{(n)}\|_\diamond = \delta_n \to 0,$$

as $n \to \infty$. We can use these coding schemes to send maximally entangled quantum states and by the Fuchs-van-de-Graaf inequalities we find that

$$F\left(\omega_2^{\otimes m_n}, \left(\mathrm{id}_2^{\otimes m_n} \otimes \left(\sum_{i=1}^{K_n} D_i^{(n)} \circ T^{\otimes n} \circ E_i^{(n)}\right)\right)\left(\omega_2^{\otimes m_n}\right)\right) \to 1,$$

as $n \to \infty$. Without loss of generality we may assume that $E_i^{(n)} \neq 0$ for each i and each n. Using the formula for the fidelity with a pure state, we find that

$$\sum_{i=1}^{K_n} p_i^{(n)} F\left(\omega_2^{\otimes m_n}, \left(\mathrm{id}_2^{\otimes m_n} \otimes D_i^{(n)} \circ T^{\otimes n} \circ \frac{E_i^{(n)}}{p_i^{(n)}}\right)\left(\omega_2^{\otimes m_n}\right)\right)^2 =: 1 - \epsilon_n \to 1,$$

as $n \to \infty$, and where we introduced

$$p_i^{(n)} = \frac{\mathrm{Tr}\left[E_i^{(n)}(\mathbb{1}_2^{\otimes m_n})\right]}{2^{m_n}} > 0.$$

Since $\sum_{i=1}^{K_n} p_i^{(n)} = 1$ for each $n \in \mathbb{N}$, there exists $i_n \in \{1, \ldots, K_n\}$ for each $n \in \mathbb{N}$ such that

$$F\left(\omega_2^{\otimes m_n}, \left(\mathrm{id}_2^{\otimes m_n} \otimes D_{i_n}^{(n)} \circ T^{\otimes n} \circ \frac{E_{i_n}^{(n)}}{p_{i_n}^{(n)}}\right)\left(\omega_2^{\otimes m_n}\right)\right)^2 \geq 1 - \epsilon_n.$$

Now, applying Lemma 3.3 we find a sequence of pure quantum states $|\psi_n\rangle \in (\mathbb{C}^2)^{\otimes m_n} \otimes \mathcal{H}_A^{\otimes n}$ such that

$$F\left(\omega_2^{\otimes m_n}, \left(\mathrm{id}_2^{\otimes m_n} \otimes D_{i_n}^{(n)} \circ T^{\otimes n}\right)(|\psi_n\rangle\langle\psi_n|)\right)^2 \to 1,$$

as $n \to \infty$. We conclude that the $D_{i_n}^{(n)}$ and $|\psi_n\rangle$ form a sequence of coding schemes for entanglement generation over T achieving the rate R. We conclude that

$$Q_{\to}(T) \leq Q_{EG}(T) = Q(T),$$

and the proof is finished. □

3.2 Classical backward communication helps!

we will now see that coding schemes allowing for classical information to be send backwards from the receiver to the sender can achieve higher communication rates than the unassisted quantum capacity. For simplicity, we will restrict to the task of entanglement generation assisted by backward communication.

Definition 3.5 (Entanglement generation schemes assisted by backward communication)**.** *Let $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ denote a quantum channel. An (n, m, δ)-coding scheme for* entanglement generation over T assisted by backward classical communication *is given by a quantum state $\rho_{RA} \in D(\mathcal{H}_R \otimes \mathcal{H}_A^{\otimes n})$ for some complex Euclidean space $\mathcal{H}_R$, an instrument $\{D_i\}_{i=1}^N$ consisting of completely positive maps*

$$D_i : B(\mathcal{H}_B^{\otimes n}) \to B((\mathbb{C}^2)^{\otimes m}),$$

such that $\sum_{i=1}^N D_i$ is a quantum channel, and quantum channels

$$E_i : B(\mathcal{H}_R) \to B((\mathbb{C}^2)^{\otimes m})$$

such that

$$F\left(\omega_2^{\otimes n}, \sum_{i=1}^N \left(E_i \otimes \left(D_i \circ T^{\otimes n}\right)\right)(\rho_{RA})\right) \geq 1 - \delta.$$

As always, we define a capacity as follows:

Definition 3.6 (Quantum capacity assisted by classical backward communication)**.** *We call a rate* $R \geq 0$ *achievable for* entanglement generation assisted by classical backward communication *over the quantum channel* $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ *if for every* $n \in \mathbb{N}$*, there exists an* (n, m_n, δ_n)*-coding scheme such that*

$$R = \lim_{n\to\infty} \frac{m_n}{n} \qquad \textit{and} \qquad \lim_{n\to\infty} \delta_n = 0.$$

The quantum capacity assisted by classical backward communication *of a quantum channel* $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ *is given by*

$$Q_{\leftarrow,EG}(T) = \sup\{R \geq 0 \; : \; R \textit{ achievable rate for quantum comm. assisted by class. backward comm.}\}.$$

Surprisingly, we have the following theorem:

Theorem 3.7. *The erasure channel* $E_\lambda : B(\mathbb{C}^2) \to B(\mathbb{C}^3)$ *with erasure probability* $\lambda \in [0,1]$ *satisfies*

$$Q_{\leftarrow,EG}(E_\lambda) \geq 1 - \lambda > 1 - 2\lambda = Q_{EG}(E_\lambda).$$

Proof. The last equality was shown in the exercises. We will now show that any rate $0 \leq R < (1-\lambda)$ is achievable for entanglement generation assisted by backward communication. The strategy to achieve such a rate is very simple: The sender just sends n halfs of maximally entangled states through $E_\lambda^{\otimes n}$. Then, the receiver identifies which of the maximally entangled states were transmitted correctly. If more than $\lfloor Rn \rfloor$ maximally entangled states have been transmitted correctly the receiver discards enough of them to be left with exactly $\lfloor Rn \rfloor$ of them. If less than $\lfloor Rn \rfloor$ maximally entangled states have been transmitted correctly the receiver outputs some fail state $\sigma_{F,n}$. Finally, the receiver communicates which tensor factors contain the $\lfloor Rn \rfloor$ maximally entangled states (if there were enough of them) and the sender restricts to those tensor factors as well. We will see that the probability of failure for this scheme approaches 0 as $n \to \infty$.

To do the above strategy formally, we will need to define an instrument and some additional quantum channels. Consider first the operator $W : \mathbb{C}^3 \to \mathbb{C}^2$ such that

$$W|i\rangle = \begin{cases} |i\rangle, & \text{if } i \in \{1,2\} \\ 0, & \text{otherwise,} \end{cases}$$

and the quantum channel $\Gamma : B(\mathbb{C}^3) \to B(\mathbb{C}^2 \otimes \mathbb{C}^2)$ given by

$$\Gamma(X) = \mathrm{Ad}_W(X) \otimes |1\rangle\langle 1| + \langle 3|X|3\rangle \frac{\mathbb{1}_2}{2} \otimes |0\rangle\langle 0|.$$

Furthermore, for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{0,1\}$ we define an operator $U_{i_1,\ldots,i_n} : (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ such that

$$U_{i_1,\ldots,i_n}(|a_1\rangle \otimes \cdots \otimes |a_n\rangle) = \bigotimes_{k: i_k = 1} |a_k\rangle \otimes \bigotimes_{l: i_l = 0} |a_l\rangle.$$

Then, we define completely positive maps $D_{i_1,\ldots,i_n} : B((\mathbb{C}^2 \otimes \mathbb{C}^2)^{\otimes n}) \to B((\mathbb{C}^2)^{\otimes \lfloor Rn \rfloor})$

$$D_{i_1,\ldots,i_n} = \begin{cases} \left(\mathrm{id}_2^{\otimes \lfloor Rn \rfloor} \otimes \mathrm{Tr}_2^{\otimes (n - \lfloor Rn \rfloor)}\right) \circ \mathrm{Ad}_{U_{i_1 \cdots i_n}} \circ \left(\mathrm{id}_2^{\otimes n} \otimes \mathrm{Ad}_{\langle i_1,\ldots,i_n|}\right), & \text{if } i_1 + \cdots + i_n \geq \lfloor Rn \rfloor, \\ \sigma_{F,n} \mathrm{Tr} \circ \left(\mathrm{id}_2^{\otimes n} \otimes \mathrm{Ad}_{\langle i_1,\ldots,i_n|}\right), & \text{otherwise,} \end{cases}$$

where $\sigma_{F,n} \in D\left((\mathbb{C}^2)^{\otimes \lfloor Rn \rfloor}\right)$ denotes some quantum state. It is easy to check that $\{D_{i_1,\ldots,i_n}\}_{i_1,\ldots,i_n}$ is an instrument. Finally, we define quantum channels $E_{i_1,\ldots,i_n} : B((\mathbb{C}^2)^{\otimes n}) \to B((\mathbb{C}^2)^{\otimes \lfloor Rn \rfloor})$ as follows: If $i_1 + \cdots + i_n \geq \lfloor Rn \rfloor$, then $E_{i_1,\ldots,i_n}$ by traces over all tensor factors k for which

$i_k = 0$ and maybe more until only $\lfloor Rn \rfloor$ tensor factors survive. If $i_1 + \cdots + i_n < \lfloor Rn \rfloor$, then $E_{i_1,\ldots,i_n}$ traces out all tensor factors and outputs $\sigma_{F,n}$ from above. Now, note that

$$(\mathrm{id}_2 \otimes \Gamma \circ E_\lambda)(\omega_2) = (1-\lambda)\omega_2 \otimes |1\rangle\langle 1| + \lambda \frac{\mathbb{1}_2}{2} \otimes \frac{\mathbb{1}_2}{2} \otimes |0\rangle\langle 0|,$$

and therefore we have

$$\sum_{i_1,\ldots,i_n} \left(E_{i_1,\ldots,i_n} \otimes (D_{i_1,\ldots,i_n} \circ (\Gamma \circ E_\lambda)^{\otimes n})\right) \left(\omega_2^{\otimes n}\right)$$
$$= p_n \omega_2^{\otimes \lfloor Rn \rfloor} + (1-p_n)\sigma_{F,n} \otimes \sigma_{F,n},$$

where

$$p_n = \mathrm{Prob}\left(X_1 + \cdots + X_n \geq \lfloor Rn \rfloor\right),$$

for some sequence of random variables $(X_k)_{k\in\mathbb{N}}$ independently and identically distributed with $\mathrm{Prob}(X_1 = 1) = \mathbb{E}\left[X_1\right] = 1-\lambda$. Since $R < 1-\lambda$ we may use the weak law of large numbers to conclude that $p_n \to 1$ as $n \to \infty$. Therefore, we have

$$F\left(\omega_2^{\lfloor Rn \rfloor}, \sum_{i_1,\ldots,i_n} \left(E_{i_1,\ldots,i_n} \otimes (D_{i_1,\ldots,i_n} \circ (\Gamma \circ E_\lambda)^{\otimes n})\right) \left(\omega_2^{\otimes n}\right)\right) \geq \sqrt{p_n} \to 1,$$

as $n \to \infty$ showing that R is achievable for entanglement generation assisted by classical backward communication.

□