Quantum information theory (MAT4430) **Spring 2021**

Lecture 4: Equivalence between open and closed quantum systems

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In the previous lecture, we have introduced the formalism of quantum theory. We started with the formalism of closed quantum systems, and to be able to consider quantum systems embedded in a larger environment we introduced the formalism of open quantum systems. In particular, we showed how objects from the formalism of open quantum systems arise from corresponding objects in the formalism of closed quantum systems. Now, we want to show thatthe two formalisms are actually equivalent. To do this, we will start with the objects in the formalism of open quantum systems and represent them using objects from the formalism of closed quantum systems. On the way, we will derive several central representation theorems for quantum channels and POVMs.

1 Operator-vector correspondence

To study the properties of bipartite quantum systems, it is surprisingly useful to note the canonical identification

$$B(\mathcal{H}_A, \mathcal{H}_B) \simeq \mathcal{H}_A \otimes \mathcal{H}_B,$$

given by the isomorphism $|\phi\rangle\langle\psi| \mapsto \overline{|\psi\rangle} \otimes |\phi\rangle$ for any $|\psi\rangle \in \mathcal{H}_A$ and $|\phi\rangle \in \mathcal{H}_B$, and extended linearly. In the following, we will call this isomorphism the *operator-vector correspondence*. We will apply it in two ways: First, we may apply it to operators in $B(\mathcal{H}_A, \mathcal{H}_B)$, and second we may apply it to operators in $B(B(\mathcal{H}_A), B(\mathcal{H}_B))$, which are linear maps from $B(\mathcal{H}_A)$ to $B(\mathcal{H}_B)$. Here, we understand $B(\mathcal{H}_A)$ and $B(\mathcal{H}_B)$ as Hilbert-Schmidt inner product spaces. Historically, the isomorphism got different names in these two settings:

Definition 1.1 (Vectorization and Choi-Jamiolkowski isomorphism)**.** *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ denote complex Euclidean spaces.*

- *The* vectorization *is the operator-vector correspondence* $\mathrm{vec} : B(\mathcal{H}_A, \mathcal{H}_B) \to \mathcal{H}_A \otimes \mathcal{H}_B$.
- *The* Choi-Jamiolkowski isomorphism *is the operator-vector correspondence*

  $$C : B(B(\mathcal{H}_A), B(\mathcal{H}_B)) \to B(\mathcal{H}_A \otimes \mathcal{H}_B),$$

  and we write $C_L := C(L)$ for linear maps $L \in B(B(\mathcal{H}_A), B(\mathcal{H}_B))$. The operator C_L is also called the Choi operator *or the* Choi matrix *of L.*

Since these isomorphisms might seem abstract, it is helpful to express them in the computational basis. For $\mathcal{H}_A = \mathbb{C}^{d_A}$ and $\mathcal{H}_B = \mathbb{C}^{d_B}$ we have

$$\mathrm{vec}\left(|i_B\rangle\langle j_A|\right) = |j_A\rangle \otimes |i_B\rangle,$$

for any $i \in \{1, \ldots, d_B\}$ and any $j \in \{1, \ldots, d_A\}$ (note that vec is antilinear in the 'A' part).

Using the lexicographic ordering of the tensor product basis, we have

$$\mathrm{vec}\,(X) = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{d_B 1} \\ x_{12} \\ \vdots \\ x_{d_B 2} \\ \vdots \end{pmatrix},$$

for any $X \in B(\mathcal{H}_A, \mathcal{H}_B)$, i.e., the vectorization operation stacks the columns of X on top of each other. In Matlab, this would correspond to using the command "reshape" transforming the matrix X to a column vector.

Similarly, we can express the Choi-Jamiolkowski isomorphism in the basis of matrix units. The matrix units on the space $B(B(\mathcal{H}_A), B(\mathcal{H}_B))$, thought of as bounded operators between Hilbert-Schmidt inner product spaces, are given by

$$\{X \mapsto |i_B\rangle\langle j_B| \operatorname{Tr}\left[(|k_A\rangle\langle l_A|)^\dagger X\right] \; : \; i,j \in \{1,\ldots,d_B\} \text{ and } k,l \in \{1,\ldots,d_A\}\}.$$

Under the Choi-Jamiolkowski isomorphism the elements of this basis transform as

$$C\left(X \mapsto |i_B\rangle\langle j_B| \operatorname{Tr}\left[(|k_A\rangle\langle l_A|)^\dagger X\right]\right) = |k_A\rangle\langle l_A| \otimes |i_B\rangle\langle j_B|,$$

for any $i,j \in \{1,\ldots,d_B\}$ and $k,l \in \{1,\ldots,d_B\}$. Therefore, given a linear map $L : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$, we have

$$C_L = \sum_{k,l=1}^{d_A} |k_A\rangle\langle l_A| \otimes L\left(|k_A\rangle\langle l_A|\right) = \begin{pmatrix} L(|1\rangle\langle 1|) & L(|1\rangle\langle 2|) & \cdots & L(|1\rangle\langle d_A|) \\ L(|2\rangle\langle d_A|) & L(|2\rangle\langle 2|) & \cdots & L(|2\rangle\langle d_A|) \\ \vdots & & \ddots & \vdots \\ L(|d_A\rangle\langle 1|) & L(|d_A\rangle\langle 2|) & \cdots & L(|d_A\rangle\langle d_A|) \end{pmatrix},$$

where we again made a choice for the ordering of the basis on the tensor product space.

There is another useful way of thinking about the canonical isomorphisms, which directly relates it to quantum theory. For this, we need to define the (unnormalized) maximally entangled quantum state:

Definition 1.2 (Maximally entangled state)**.** *The vector*

$$|\Omega_d\rangle = \mathrm{vec}\,(\mathbb{1}_{\mathbb{C}^d}) = \sum_{i=1}^{d} |i\rangle \otimes |i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d,$$

is called the (unnormalized) maximally entangled state*. We will write*

$$\omega_d = |\Omega_d\rangle\langle\Omega_d| \in B(\mathbb{C}^d)^+.$$

Now, the following lemma is easy to see (and you should check this yourself):

Lemma 1.3 (Operator-vector correspondence via maximally entangled state)**.** *Consider complex Euclidean spaces $\mathcal{H}_A = \mathbb{C}^{d_A}$ and $\mathcal{H}_B = \mathbb{C}^{d_B}$, a linear map $L : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and an operator $X \in B(\mathcal{H}_A, \mathcal{H}_B)$. Then, we have*

$$\mathrm{vec}\,(X) = (\mathbb{1}_{\mathcal{H}_A} \otimes X)\,|\Omega_{d_A}\rangle,$$

and

$$C_L = (\mathrm{id}_A \otimes L)\,(\omega_{d_A}).$$

The next lemma proves two identities for the canonical isomorphisms, which will turn out to be extremely useful:

Lemma 1.4 (Necklace identities)**.** *For complex Euclidean spaces $\mathcal{H}_A = \mathbb{C}^{d_A}$ and $\mathcal{H}_B = \mathbb{C}^{d_B}$ we have the following identities:*

1. *For any operator $E \in B(\mathcal{H}_A, \mathcal{H}_B)$ we have*

$$(\mathbb{1}_{\mathcal{H}_A} \otimes E) |\Omega_{d_A}\rangle = \left(E^T \otimes \mathbb{1}_{\mathcal{H}_B}\right) |\Omega_{d_B}\rangle.$$

2. *For any linear map $L : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ satisfying $L(X)^\dagger = L(X^\dagger)$ for every $X \in B(\mathcal{H}_A)$ we have*

$$(\mathrm{id}_A \otimes L)(\omega_{d_A}) = (\vartheta_A \circ L^* \circ \vartheta_B \otimes \mathrm{id}_B)(\omega_{d_B}),$$

where $\vartheta_A : B(\mathcal{H}_A) \to B(\mathcal{H}_A)$ and $\vartheta_B : B(\mathcal{H}_B) \to B(\mathcal{H}_B)$ denote the transpose maps in the computational bases of the respective spaces, i.e., $\vartheta_A(X) = X^T$ for $X \in B(\mathcal{H}_A)$.

The name "necklace identities" comes from a certain graphical calculus that can be used to better understand matrix and vector operations in multilinear settings: In this calculus all of the basic objects vectors, matrices and higher-order tensors are represented as boxes with "legs" corresponding to indices that can be summed over. For example, a matrix $A \in \mathcal{M}_d$ with entries A_{ij} has two indices i and j and is therefore represented as a box with two legs. We can then join the legs of two boxes to indicate a contraction, i.e., a sum over the entries of the objects where the joined indices take the same value. Figure 3 introduces the basic elements of this way of thinking. Using the graphical calculus, we can write the first necklace identity as in Figure 1.

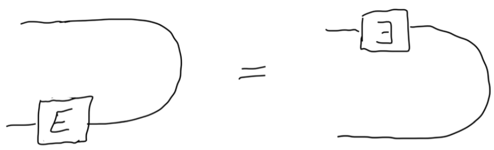

Figure 1: First necklace identity.

Another useful observation is, that the canonical isomorphism defines an isometry between different bipartite Euclidean spaces and Hilbert-Schmidt inner product spaces.

Lemma 1.5 (Isometry)**.** *For any $X, Y \in B(\mathcal{H}_A, \mathcal{H}_B)$ we have*

$$\langle \mathrm{vec}(X), \mathrm{vec}(Y)\rangle = \mathrm{Tr}\left[X^\dagger Y\right].$$

In particular, the operator-vector correspondence is an isometric isomorphism between the complex Euclidean space $\mathcal{H}_A \otimes \mathcal{H}_B$ and the Hilbert-Schmidt inner product space over $B(\mathcal{H}_A, \mathcal{H}_B)$.

The proof of this lemma can be done using the graphical calculus as in Figure 2. Alternatively, the lemma can also be proved by expanding all operators in the computational basis. If you want to check the lemma that way, please do so.

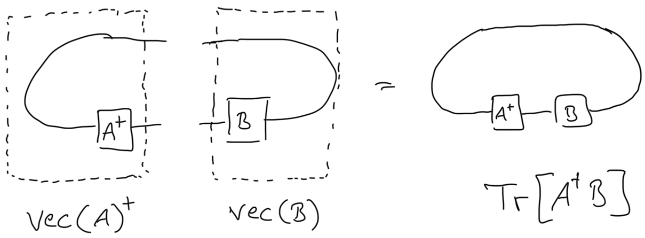

Figure 2: Proof by picture.

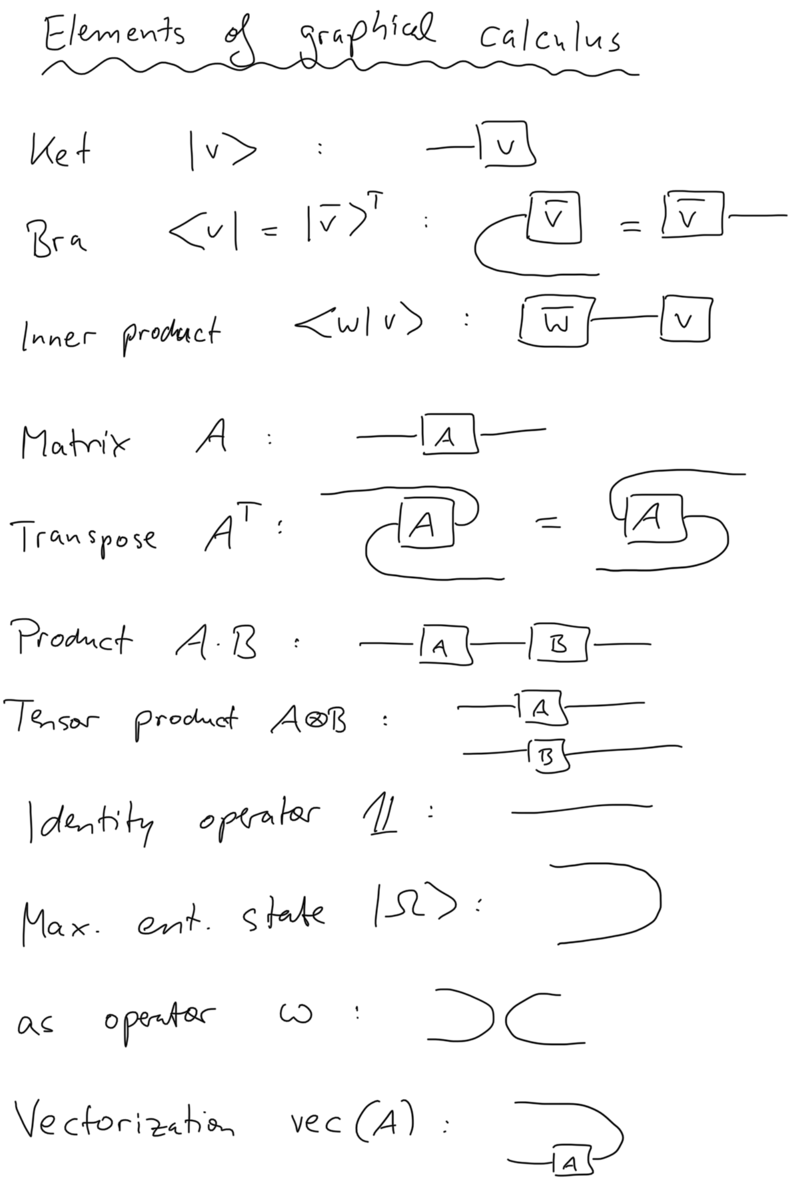

Figure 3: Graphical calculus.

2 Choi-Kraus representation of completely positive maps

For any operator $K \in B(\mathcal{H}_A, \mathcal{H}_B)$, we define a linear map $\mathrm{Ad}_K : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ by

$$\mathrm{Ad}_K(X) = KXK^\dagger.$$

Note that Ad_K is positive for any $K \in B(\mathcal{H}_A, \mathcal{H}_B)$ since

$$\mathrm{Ad}_K(YY^\dagger) = KYY^\dagger K^\dagger = (KY)(KY)^\dagger,$$

and Ad_K is even completely positive since $\mathrm{id}_E \otimes \mathrm{Ad}_K = \mathrm{Ad}_{\mathbb{1}_{\mathcal{H}_E} \otimes K}$ for any Euclidean space $\mathcal{H}_E$. We will now show that the maps Ad_K can be used to express general completely positive maps and quantum channels.

Recall the canonical Choi-Jamiolkowski isomorphism $C : B(B(\mathcal{H}_A), B(\mathcal{H}_B)) \to B(\mathcal{H}_A \otimes \mathcal{H}_B)$ associating a linear map $L : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ to its Choi matrix C_L. We have:

Theorem 2.1 (Choi and Kraus)**.** *For a linear map $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ the following are equivalent:*

1. *The map T is completely positive.*
2. *The Choi matrix C_T is positive.*
3. *There exist operators $\{K_i\}_{i=1}^R \subset B(\mathcal{H}_A, \mathcal{H}_B)$ and some $R \in \mathbb{N}$ such that*

$$T = \sum_{i=1}^{R} \mathrm{Ad}_{K_i}\,. \tag{1}$$

Moreover, we can always choose $R = \mathrm{rk}\,(C_T)$ in 3. *above.*

Proof. To show that 1. implies 2. we use Lemma 1.3 to conclude that

$$C_T = (\mathrm{id}_A \otimes T)\,(\omega_{d_A}) \geqslant 0$$

for any completely positive map T, since $\omega_{d_A} \in B\,(\mathcal{H}_A \otimes \mathcal{H}_A)^+$. To show that 2. implies 3. we apply the spectral decomposition to the Hermitian operator C_T. We have

$$C_T = \sum_{i=1}^{R} |\psi_i\rangle\langle\psi_i|,$$

for $R = \mathrm{rk}\,(C_T)$ and some (unnormalized) vectors $|\psi_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Using the inverse of the vectorization isomorphism and Lemma 1.3 we have

$$|\psi_i\rangle = \mathrm{vec}\,(K_i) = (\mathbb{1}_{\mathcal{H}_A} \otimes K_i)\,|\Omega_{d_A}\rangle,$$

for some operators $K_i \in B(\mathcal{H}_A, \mathcal{H}_B)$. Combining the previous equations, we find that

$$C_T = \sum_{i=1}^{R} (\mathbb{1}_{d_A} \otimes K_i)\,\omega_{d_A}\,(\mathbb{1}_{d_A} \otimes K_i)^\dagger = \sum_{i=1}^{R} C_{\mathrm{Ad}_{K_i}} = C_{\sum_{i=1}^{R} \mathrm{Ad}_{K_i}}.$$

Applying the inverse of the Choi-Jamiolkowski isomorphism finishes the proof. Showing that 3. implies 1. is easy since each summand in (1) (and hence the whole sum) is completely positive. $\square$

The decomposition (1) is called the Choi decomposition in the mathematics literature, and the Kraus decomposition in the physics literature[1]. Throughout this course, we will call it the *Choi-Kraus decomposition* as a compromise. It should be noted that this decomposition is not unique in general.

Theorem 2.2. *Let $\mathcal{H}$ denote a complex Euclidean space and consider sets of operators $\{K_n\}_{n=1}^N \subset B(\mathcal{H})$ and $\{L_m\}_{m=1}^M \subset B(\mathcal{H})$. The following are equivalent:*

1. *We have*
$$\sum_{n=1}^{N} \mathrm{Ad}_{K_n} = \sum_{m=1}^{M} \mathrm{Ad}_{L_m} .$$
2. *There exists a unitary $U \in \mathcal{U}\left(\mathbb{C}^R\right)$ for $R = \max(M, N)$ such that*
$$K_n = \sum_{m=1}^{R} U_{nm} L_m .$$

Here the smaller of the two sets is extended by zero operators to make them the same size.

Proof. By the Choi-Jamiolkowski isomorphism this theorem is equivalent to the statement proved in Exercise 4 on Sheet 2. □

As a corollary we obtain the following characterization of quantum channels:

Corollary 2.3 (Characterization of quantum channels)**.** *A linear map $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is a quantum channel if and only if it can be written as*
$$T = \sum_{i=1}^{R} \mathrm{Ad}_{K_i},$$
for operators $\{K_i\}_{i=1}^R \subset B(\mathcal{H}_A, \mathcal{H}_B)$ satisfying
$$\sum_{i=1}^{R} K_i^\dagger K_i = \mathbb{1}_{\mathcal{H}_A}. \tag{2}$$

Proof. By Theorem 2.1, we only have to check that (2) holds if and only if the map T is trace-preserving. Indeed, this is easy to check since
$$\mathrm{Tr}\left[\sum_{i=1}^{R} K_i X K_i^\dagger\right] = \mathrm{Tr}\left[\left(\sum_{i=1}^{R} K_i^\dagger K_i\right) X\right],$$
for any $X \in B(\mathcal{H}_A)$ by cyclicity of the trace. □

If you are reading these lecture notes in order, then you might remember that the Choi-Kraus decomposition has already appeared. When computing a concrete example of an instrument in the third lecture, it appeared in the definition of the linear maps T_n. In light of the previous discussion, the general definition of instruments should feel more natural now:

Definition 2.4 (Instruments)**.** *A set $\{T_n\}_{n=1}^N$ of completely positive maps $T_n : B(\mathcal{H}_A) \to \mathcal{H}_B$ is called an* instrument *if the sum $T = \sum_{n=1}^N T_n$ is a quantum channel.*

In this general form, instruments contain both quantum channels (if $N = 1$) and POVMs (if $\dim(\mathcal{H}_B) = 1$) as special cases. We could therefore choose to formulate quantum theory in terms of instruments alone. However, we will not do so. Historically, quantum information theory was build on the notions of quantum channels and POVMs, and for many applications this is sufficient and considerably simpler. Therefore, we will put emphasize on these concepts and we keep general instruments in mind for when they are really neccessary.

[1] Historically, it was discovered by Choi, but popularized and extended by Kraus.

3 Equivalence of closed and open quantum formalism

In the third lecture we have seen how certain pure quantum states, time-evolutions and PVMs acting on a composite quantum system 'AE' give rise to quantum states, quantum channels and POVMs when restricted to the subsystem 'A'. Now, we will go in the opposite direction and show that every quantum state, quantum channel and POVM acting on a quantum system 'A' arises from the corresponding type of object acting on some composite system 'AE'. This implies that the postulates of closed quantum systems and the postulates of open quantum systems are actually equivalent. Mathematically, the extension of an object to a larger object with some additional structure is called a *dilation*, and the theorems proved in this section give some examples of this general principle.

Purifications of quantum states: Is any quantum state the reduced density operator of some pure state? To answer this question, we will need the following definition:

Definition 3.1 (Purification). *A pure quantum state* $|\psi_{AB}\rangle\langle\psi_{AB}| \in \text{Proj}\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$ *is called the purification of a quantum state* $\rho_A \in D\left(\mathcal{H}_A\right)$ *if we have*

$$\rho_A = \text{Tr}_B\left[|\psi_{AB}\rangle\langle\psi_{AB}|\right].$$

We will now show that any quantum state has a purification. For this we will need the canonical isomorphism vec : $\mathcal{H}_A \otimes \mathcal{H}_B \to B(\mathcal{H}_A, \mathcal{H}_B)$ introduced in the beginning of this lecture. We have the following theorem:

Theorem 3.2 (Existence of purifications). *For a density operator* $\rho_A \in D\left(\mathcal{H}_A\right)$ *consider the pure quantum state* $|\psi_{A'A}\rangle\langle\psi_{A'A}| \in \text{Proj}\left(\mathcal{H}_A \otimes \mathcal{H}_A\right)$ *given by*

$$|\psi_{A'A}\rangle = \text{vec}\left(\sqrt{\rho_A}\right) \in \mathcal{H}_A \otimes \mathcal{H}_A,$$

where $\sqrt{\rho_A}$ *denotes the unique positive square root of the positive operator* ρ*. Then, we have*

$$\rho_A = \left(\text{Tr}_{A'} \otimes \text{id}_A\right)\left[|\psi_{A'A}\rangle\langle\psi_{A'A}|\right],$$

showing that $|\psi_{A'A}\rangle\langle\psi_{A'A}|$ *is a purification of* ρ*.*

Proof. By Lemma 1.5, we have

$$\langle\psi_{AA'}|\psi_{AA'}\rangle = \langle\sqrt{\rho_A}, \sqrt{\rho_A}\rangle_{HS} = \text{Tr}\left[\rho_A\right] = 1.$$

This shows that $|\psi_{AA'}\rangle\langle\psi_{AA'}|$ is a pure quantum state. Using Lemma 1.3, we have

$$\left(\text{Tr}_{A'} \otimes \text{id}_A\right)\left[|\psi_{A'A}\rangle\langle\psi_{A'A}|\right] = \left(\text{Tr}_{A'} \otimes \text{Ad}_{\sqrt{\rho}}\right)\left(\omega_{d_A}\right) = \sqrt{\rho_A}\mathbb{1}_{\mathcal{H}_A}\sqrt{\rho_A} = \rho_A,$$

since

$$\left(\text{Tr}_{A'} \otimes \text{id}_A\right)\left(\omega_{d_A}\right) = \mathbb{1}_{\mathcal{H}_A}.$$

This finishes the proof. □

The previous theorem shows how to relate each quantum state to a pure state by vectorizing a matrix. Since the map vec : $\mathcal{H}_A \otimes \mathcal{H}_B \to B(\mathcal{H}_A, \mathcal{H}_B)$ is an isomorphism, we can also reverse these ideas and go from vectors to matrices: Starting with a pure quantum state $|\psi_{AB}\rangle\langle\psi_{AB}| \in \text{Proj}\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$, we have

$$|\psi_{AB}\rangle = \text{vec}\left(X_\psi\right),$$

for some operator $X_\psi \in B(\mathcal{H}_A, \mathcal{H}_B)$ such that the reduced density operators $\rho_A = \mathrm{Tr}_B\,[|\psi_{AB}\rangle\langle\psi_{AB}|]$ and $\rho_B = \mathrm{Tr}_A\,[|\psi_{AB}\rangle\langle\psi_{AB}|]$ can be expressed as

$$\rho_A = X_\psi^T \overline{X_\psi} \quad \text{and} \quad \rho_B = X_\psi X_\psi^\dagger.$$

Let $d_A = \dim(\mathcal{H}_A)$ and $d_B = \dim(\mathcal{H}_B)$ denote the dimensions of the involved Euclidean spaces. When $d_A = d_B$, the previous theorem shows that we can choose $X_\psi = \sqrt{\rho_B}$. Even if $d_A \neq d_B$ there is a close connection between purifications and reduced density operators. Next, we will express this in a concrete form: Using the singular value decomposition, we can then find orthonormal bases $\{|\overline{a_i}\rangle\}_{i=1}^{d_A}$ and $\{|b_i\rangle\}_{i=1}^{d_B}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that

$$X_\psi = \sum_{i=1}^{\min(d_A,d_B)} \sqrt{\lambda_i}|b_i\rangle\langle\overline{a_i}|,$$

where we have denoted the singular values of X_ψ by $\sqrt{\lambda_i} \in \mathbb{R}^+$ for reasons that will become clear in a moment. Since vectorization is linear, we have the following theorem:

Theorem 3.3 (Schmidt decomposition)**.** *Consider Euclidean spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with $\dim(\mathcal{H}_A) = d_A$ and $\dim(\mathcal{H}_B) = d_B$. For any pure state $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ there exist orthonormal bases $\{|a_i\rangle\}_{i=1}^{d_A}$ and $\{|b_i\rangle\}_{i=1}^{d_B}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, such that*

$$|\psi_{AB}\rangle = \sum_{i=1}^{\min(d_A,d_B)} \sqrt{\lambda_i}|a_i\rangle \otimes |b_i\rangle,$$

for numbers $\sqrt{\lambda_i} \in \mathbb{R}^+$ called the Schmidt coefficients *of $|\psi_{AB}\rangle$. The number of non-zero Schmidt coefficients is called the* Schmidt rank *of the pure state $|\psi_{AB}\rangle$.*

The Schmidt decomposition makes the correspondence between purifications and reduced density operators quite clear. In particular, we have the following:

- The rank of a density operator equals the Schmidt rank of its purification, and the Schmidt rank of a pure state equals the rank of its reduced density operators.

- The minimal dimension of a Euclidean space $\mathcal{H}_E$ such that there is a purification $|\psi_{AE}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_E$ of a given density operator $\rho_A \in D(\mathcal{H}_A)$ equals $\mathrm{rk}(\rho_A)$.

- Any purification $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ of a density operator $\rho_A \in D(\mathcal{H}_A)$ arises from a purification $|\psi_{AE}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_E$ for $d_E = \mathrm{rk}(\rho_A)$ as $|\psi_{AB}\rangle = (\mathbb{1}_{\mathcal{H}_A} \otimes V)|\psi_{AE}\rangle$ for an isometry $V : \mathcal{H}_E \to \mathcal{H}_B$.

- The eigenvalues of a density operator equals the square roots of the Schmidt coefficients of its purification, and squaring the Schmidt coefficients of a pure state gives the eigenvalues of its reduced density operators.

- In particular, the eigenvalues of the two reduced density operators of the same bipartite pure state coincide.

The last of these facts will turn out to be of particular importance later. We have shown that every density operator arises as a reduced density operator of some pure state, and that every pure state arises as the purification of some density operator. We will now see how to find similar correspondences between the other objects of the formalisms of closed and open quantum systems.

The Naimark and Stinespring dilations: The following theorem is the foundation of the dilation theorems of both POVMs and quantum channels:

Theorem 3.4 (Naimark-Stinespring dilation)**.** *For any set $\{K_n\}_{n=1}^N \subset B(\mathcal{H}_A, \mathcal{H}_B)$ satisfying $\sum_{n=1}^N K_n^\dagger K_n = \mathbb{1}_{\mathcal{H}_A}$, there exists a Euclidean space $\mathcal{H}_E$ with dimension $d_E = N$ and an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that the following two statements hold:*

1. *For any $m \in \{1, \dots, N\}$ we have*
$$V^\dagger \left(\mathbb{1}_{\mathcal{H}_B} \otimes |m\rangle\langle m|_E\right) V = K_m^\dagger K_m.$$

2. *We have*
$$\sum_{n=1}^N K_n X K_n^\dagger = \mathrm{Tr}_E \left[V X V^\dagger\right],$$
for any $X \in B(\mathcal{H}_A)$.

In the first case, the isometry V is called the Naimark dilation *of the POVM $\{K_n^\dagger K_n\}_{n=1}^N$ and in the second case it is called the Stinespring dilation of the quantum channel with Kraus operators $\{K_n\}_{n=1}^N$.*

Proof. We set $\mathcal{H}_E = \mathbb{C}^N$ and define an operator $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ by
$$V = \sum_{n=1}^N K_n \otimes |n\rangle_E.$$
Since $\sum_{n=1}^N K_n^\dagger K_n = \mathbb{1}_{\mathcal{H}_A}$ the operator V defines an isometry and the two statements in the theorem can be checked easily. □

The dilations in the previous theorem have neat physical interpretations: For a set of operators $\{K_n\}_{n=1}^N \subset B(\mathcal{H}_A, \mathcal{H}_B)$ satisfying $\sum_{n=1}^N K_n^\dagger K_n = \mathbb{1}_{\mathcal{H}_A}$ we denote by $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ the quantum channel with $T(X) = \sum_{n=1}^N K_n X K_n^\dagger$ for any $X \in B(\mathcal{H}_A)$, and by $\{Q_n\}_{n=1}^N \subset B(\mathcal{H}_A)^+$ the POVM given by $Q_n = K_n^\dagger K_n$.

- If the Euclidean space $\mathcal{H}_E$ in the Stinespring dilation $T(X) = \mathrm{Tr}_E \left[V X V^\dagger\right]$ of a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ has dimension $d_E > 1$, then it means that some information about the system 'A' got leaked to the environment system 'E'.
- We can implement the measurement of $\{Q_n\}_{n=1}^N \subset B(\mathcal{H}_A)^+$ on any quantum state $\rho_A \in D(\mathcal{H}_A)$ by the following process: First, we apply the quantum channel $X \mapsto \mathrm{Tr}_B \left[V X V^\dagger\right]$ to the state ρ_A and then we measure the outcome using the PVM $\{|n\rangle\langle n|_E\}_{n=1}^N$.

To make this a bit more precise, we can state the following definition:

Definition 3.5 (Complementary channel)**.** *If the isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ is the Stinespring dilation of a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$, then the complementary channel $T^c : B(\mathcal{H}_A) \to B(\mathcal{H}_E)$ is given by*
$$T^c(X) = \mathrm{Tr}_B \left(V X V^\dagger\right).$$

The complementary channel describes the channel as it is 'seen by the environment'. Combining, the two points from above, we get the following physical interpretation of instruments: Define the instrument $\{T_n\}_{n=1}^N$ by $T_n(X) = K_n X K_n^\dagger$ for any $X \in B(\mathcal{H}_A)$ such that $T = \sum_{n=1}^N T_n$. The POVM performed by this instrument is exactly the PVM $\{|n\rangle\langle n|_E\}_{n=1}^N$

on the output of the complementary channel T^c, and if we forget (or do not know) the outcome of this measurement the final quantum state can be obtained by applying the channel T to the input.

We will later express many information-processing tasks in terms of the channel and its complementary channel together. Intuitively, this becomes clear when considering a cryptographic protocol involving the quantum channel T: Any information leaked to the environment (i.e., obtained by applying the complementary channel to the data) could in principle be retrieved by some eavesdropper. Private communication protocols, therefore, have to use clever encodings to make the complementary channel as noisy as possible, so that no useful information can be extracted from it.

Quantum channels and POVMs as interactions with the environment: Does any quantum channel arise as the reduction of some unitary time-evolution, and does any POVM arise from some PVM? In this section, we will answer these two questions in the affirmative. For this, we will apply the Naimark-Stinespring dilation theorem and a trick of how to embed an isometry into a unitary operator.

To explain the trick, consider an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ and pure quantum states $|\psi_{BE}\rangle\langle\psi_{BE}| \in \text{Proj}\,(\mathcal{H}_B \otimes \mathcal{H}_E)$ and $|\phi_A\rangle\langle\phi_A| \in \text{Proj}\,(\mathcal{H}_A)$. Now, define two sets $\{|x_i\rangle\}_{i=1}^{d_A}, \{|y_i\rangle\}_{i=1}^{d_A} \subset \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E$ of orthogonal vectors by setting

$$|x_i\rangle = |\phi_A\rangle \otimes V|i\rangle_A,$$

and

$$|y_i\rangle = |i\rangle_A \otimes |\psi_{BE}\rangle.$$

Since the two sets of orthogonal vectors have the same size, there exists a unitary operator $U \in \mathcal{U}\,(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ such that

$$U|y_i\rangle = |x_i\rangle \text{ for any } i \in \{1, \ldots, d_A\},$$

and we can obtain the isometry V by

$$(\langle\phi_A| \otimes \mathbb{1}_{\mathcal{H}_{BE}})\, U\,(|x\rangle_A \otimes |\psi_{BE}\rangle) = V|x\rangle. \tag{3}$$

We will use this trick in two instances to express quantum channels and POVMs as reductions of unitary time-evolutions and PVMs, respectively. We start with quantum channels:

Theorem 3.6 (Open system representation of quantum channels)**.** *Consider a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and denote by $\mathcal{H}_E$ the Euclidean space appearing in its Stinespring dilation. For any pure state $|\psi_{BE}\rangle\langle\psi_{BE}| \in \text{Proj}\,(\mathcal{H}_B \otimes \mathcal{H}_E)$ there exists a unitary $U \in \mathcal{U}\,(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ such that*

$$T(X) = \text{Tr}_{AE}\left(U\,(X \otimes |\psi_{BE}\rangle\langle\psi_{BE}|)\,U^\dagger\right),$$

for any $X \in B\,(\mathcal{H}_A)$.

Proof. By Theorem 3.4, we can consider the Stinespring dilation $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that $T(X) = \text{Tr}_E\left(VXV^\dagger\right)$ for any $X \in B\,(\mathcal{H}_A)$. Let $U \in \mathcal{U}\,(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_E)$ denote a unitary operator such that (3) holds with $|\psi_{BE}\rangle$ and some pure state $|\phi_A\rangle\langle\phi_A| \in \text{Proj}\,(\mathcal{H}_A)$. It is then easy to check that

$$\text{Tr}_{AE}\left[U\,(|i\rangle\langle j|_A \otimes |\psi_{BE}\rangle\langle\psi_{BE}|)\,U^\dagger\right] = \text{Tr}_{AE}\left[|\phi_A\rangle\langle\phi_A| \otimes V|i\rangle\langle j|_A V^\dagger\right] = T\,(|i\rangle\langle j|_A)\,,$$

for any $i, j \in \{1, \ldots, d_A\}$ and the proof is finished by linearity.

□

The previous theorem shows that any quantum channel can be written in the concrete form introduced in Lecture 3 for some unitary time-evolution and some environment state. Next, we show the same statement for POVMs:

Theorem 3.7 (Environment induced measurements)**.** *Consider a POVM* $\{Q_i\}_{n=1}^N \subset B(\mathcal{H}_A)^+$ *and denote by* $\mathcal{H}_E$ *the Euclidean space appearing in its Naimark dilation. For any pure state* $|\psi_E\rangle\langle\psi_E| \in \mathrm{Proj}(\mathcal{H}_E)$ *there exists a PVM* $\{P_n\}_{n=1}^N \subset \mathrm{Proj}(\mathcal{H}_A \otimes \mathcal{H}_E)$ *such that*

$$\mathrm{Tr}[Q_n \rho_A] = \mathrm{Tr}[P_n(\rho_A \otimes |\psi_E\rangle\langle\psi_E|)],$$

for any $n \in \{1, \ldots, N\}$ *and any* $\rho_A \in D(\mathcal{H}_A)$ *.*

Proof. By Theorem 3.4, we can consider the Naimark dilation $V : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_E$ such that

$$Q_n = V^\dagger (\mathbb{1}_{\mathcal{H}_A} \otimes |n\rangle\langle n|_E) V,$$

for any $n \in \{1, \ldots, N\}$. Let $U \in \mathcal{U}(\mathcal{H}_A \otimes \mathcal{H}_E)$ denote a unitary operator such that (3) holds with the pure state $|\psi_E\rangle\langle\psi_E| \in \mathrm{Proj}(\mathcal{H}_E)$ (where the system 'B' has dimension 1), i.e., such that

$$U(|x_A\rangle \otimes |\psi_E\rangle) = V|x_A\rangle,$$

for any $|x_A\rangle \in \mathcal{H}_A$. For each $n \in \{1, \ldots, N\}$ we define the projector

$$P_n = U^\dagger (\mathbb{1}_{\mathcal{H}_A} \otimes |n\rangle\langle n|_E) U.$$

Then, we compute

$$\begin{aligned}
\mathrm{Tr}[P_n(\rho_A \otimes |\psi_E\rangle\langle\psi_E|)] &= \mathrm{Tr}\left[(\mathbb{1}_{\mathcal{H}_A} \otimes |n\rangle\langle n|_E) U (\rho_A \otimes |\psi_E\rangle\langle\psi_E|) U^\dagger\right] \\
&= \mathrm{Tr}\left[(\mathbb{1}_{H_A} \otimes |n\rangle\langle n|_E) V \rho_A V^\dagger\right] \\
&= \mathrm{Tr}[Q_n \rho_A].
\end{aligned}$$

It is easy to check that $\sum_{n=1}^N P_n = \mathbb{1}_{\mathcal{H}_A} \otimes \mathbb{1}_{\mathcal{H}_E}$, and the proof is finished. □