Quantum information theory (MAT4430) **Spring 2021**

Lecture 6: Bipartite entanglement (Part II)

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In the previous lectures, we have introduced different cones in the real vector space $B(\mathcal{H}_A \otimes \mathcal{H}_B)_{sa}$ which play a role in the theory of quantum entanglement. Specifically, these were the

- **Separable cone:**

$$\text{Sep}\left(\mathcal{H}_A, \mathcal{H}_B\right) = \text{cone}\{\rho_A \otimes \sigma_B \ : \ \rho_A \in D(\mathcal{H}_A), \sigma_B \in D(\mathcal{H}_B)\}.$$

- **Block-positive cone:**

$$\text{BP}\left(\mathcal{H}_A, \mathcal{H}_B\right) = \{C_P \ : \ P : B(\mathcal{H}_A) \to B(\mathcal{H}_B) \text{ positive}\}.$$

- **Cone of positive semidefinite matrices:**

$$B\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)^+ = \{X_{AB} \text{ positive semidefinite}\}.$$

- **Positive partial transpose cone:**

$$\text{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right) = \{X_{AB} \ : \ X_{AB}^\Gamma = \left(\text{id}_A \otimes \vartheta_B\right)\left(X_{AB}\right) \in B\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)^+\}.$$

Two other cones can be formed as the intersection

$$B\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)^+ \cap \text{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right),$$

and the join (or sum)

$$B\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)^+ \vee \text{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right).$$

We will now continue our study of these cones and in particular focus on the previous two cases.

1 Cones, duality and classes of linear maps

To study these cones, we will need the following lemma:

Lemma 1.1 (Dual of positive PPT operators)**.** *For complex Euclidean spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ the cone $PPT(\mathcal{H}_A, \mathcal{H}_B)$ is selfdual and we have*

$$\left(PPT(\mathcal{H}_A, \mathcal{H}_B) \cap B(\mathcal{H}_A \otimes \mathcal{H}_B)^+\right)^* = B(\mathcal{H}_A \otimes \mathcal{H}_B)^+ \vee PPT(\mathcal{H}_A, \mathcal{H}_B).$$

Proof. To show that $\left(\text{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right)\right)^* = \text{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right)$, we note that

$$\langle Y, X\rangle_{HS} = \text{Tr}\left[YX\right] = \text{Tr}\left[Y^\Gamma X^\Gamma\right] = \langle Y^\Gamma, X^\Gamma\rangle_{HS},$$

where we used that the transpose map is selfadjoint and its own inverse. We conclude that $Y \in B\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)_{sa}$ satisfies $\langle Y, X\rangle \geqslant 0$ for any $X \in \text{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right)$ if and only if Y^Γ satisfies $\langle Y^\Gamma, X^\Gamma\rangle \geqslant 0$ for any $X^\Gamma \in B\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)^+$. But this holds if and only if $Y^\Gamma \in B\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)^+$ which is equivalent to $Y \in \text{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right)$.

By elementary properties of duals we have

$$\left(\mathrm{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right) \cap B(\mathcal{H}_A \otimes \mathcal{H}_B)^+\right)^* = \overline{B(\mathcal{H}_A \otimes \mathcal{H}_B)^+ \vee \mathrm{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right)}.$$

To show that the closure is not needed, we consider any $X \in B(\mathcal{H}_A \otimes \mathcal{H}_B)^+ \vee \mathrm{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right)$, which we can write as $X = X_1 + X_2$ with $X_1 \in B(\mathcal{H}_A \otimes \mathcal{H}_B)^+$ and $X_2 \in \mathrm{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right)$. Since $\mathrm{Tr}\left[X_1\right] > 0$ and $\mathrm{Tr}\left[X_2\right] > 0$ (the transposition is trace-preserving), we can normalize the trace of X and find that

$$\frac{X}{\mathrm{Tr}\left[X\right]} = \frac{\mathrm{Tr}\left[X_1\right]}{\mathrm{Tr}\left[X\right]} \frac{X_1}{\mathrm{Tr}\left[X_1\right]} + \frac{\mathrm{Tr}\left[X_2\right]}{\mathrm{Tr}\left[X\right]} \frac{X_2}{\mathrm{Tr}\left[X_2\right]}.$$

Since $\mathrm{Tr}\left[X\right] = \mathrm{Tr}\left[X_1\right] + \mathrm{Tr}\left[X_2\right]$, we conclude that

$$\frac{X}{\mathrm{Tr}\left[X\right]} \in \mathrm{conv}\left(D\left(\mathcal{H}_A \otimes \mathcal{H}_B\right) \cup D^\Gamma\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)\right),$$

with

$$D^\Gamma\left(\mathcal{H}_A \otimes \mathcal{H}_B\right) = \mathrm{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right) \cap \{Y \in B(\mathcal{H}_A \otimes \mathcal{H}_B)_{sa} \ : \ \mathrm{Tr}\left[Y\right] = 1\}.$$

Since both $D\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$ and $D^\Gamma\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$ are compact, we conclude that

$$B = \mathrm{conv}\left(D\left(\mathcal{H}_A \otimes \mathcal{H}_B\right) \cup D^\Gamma\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)\right),$$

is compact as the convex hull of a compact set. By the previous argument, it is a compact base of $B(\mathcal{H}_A \otimes \mathcal{H}_B)^+ \vee \mathrm{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right)$. This shows that this join of the two cones is closed and the proof is finished. □

Now, we can summarize all inclusions and dualities between these cones as in Figure 1.

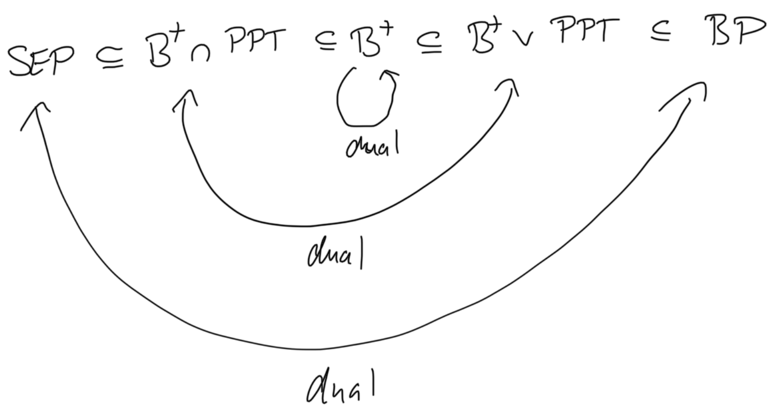

Figure 1: Inclusions and dualities of the cones introduced so far.

Among the cones introduced above, some of them were given as Choi matrices of certain classes of linear maps. For instance, the block-positive cone was defined as the cone of Choi matrices of positive maps, and we have seen before that the cone of positive semidefinite matrices arises as the cone of Choi matrices of completely positive maps. These are no coincidences! By the Choi-Jamiolkowski isomorphism any subcone of $B(\mathcal{H}_A \otimes \mathcal{H}_B)_{sa}$ defines a subcone of Hermiticity preserving linear maps $P : \mathcal{H}_A \to \mathcal{H}_B$ (which in the following cases will always be positive). Such a map is called

- *entanglement breaking* if $C_P \in \mathrm{Sep}\left(\mathcal{H}_A, \mathcal{H}_B\right)$.

- *completely positive (CP)* if $C_P \in B(\mathcal{H}_A \otimes \mathcal{H}_B)^+$.
- *completely copositive (coCP)* if $C_P \in \mathrm{PPT}(\mathcal{H}_A, \mathcal{H}_B)$, which is equivalent to $\vartheta_B \circ P$ being CP.
- *decomposable* if $C_P \in B(\mathcal{H}_A \otimes \mathcal{H}_B)^+ \vee \mathrm{PPT}(\mathcal{H}_A, \mathcal{H}_B)$, which is equivalent to $P = T_1 + \vartheta_B \circ T_2$ for T_1, T_2 CP.
- *positive* if $C_P \in \mathrm{BP}(\mathcal{H}_A, \mathcal{H}_B)$.

We finish this section with the following lemma characterizing the cone $\mathrm{BP}(\mathcal{H}_A, \mathcal{H}_B)$ without explicitly using positive maps:

Lemma 1.2 (Characterizing block-positivity)**.** *For complex Euclidean spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ consider an operator $X_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)_{sa}$. The following are equivalent:*

1. *We have $X_{AB} \in \mathrm{BP}(\mathcal{H}_A, \mathcal{H}_B)$.*
2. *We have*
$$(\langle x| \otimes \langle y|)\, X_{AB}\, (|x\rangle \otimes |y\rangle) \geqslant 0,$$
for every $|x\rangle \in \mathcal{H}_A$ and every $|y\rangle \in \mathcal{H}_B$.

Proof. Consider the Hermiticity-preserving map $Q : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ satisfying $X_{AB} = C_Q$, which exists by the Choi-Jamiolkowski isomorphism. The equivalence of the two statements follows from
$$(\langle x| \otimes \langle y|)\, X_{AB}\, (|x\rangle \otimes |y\rangle) = \langle y|Q(|\overline{x}\rangle\langle\overline{x}|)|y\rangle,$$
which holds for every $|x\rangle \in \mathcal{H}_A$ and every $|y\rangle \in \mathcal{H}_B$ by using the necklace identities. The map Q is positive if and only if the right-hand side in the previous equality is non-negative. ☐

2 When are two qubits entangled?

We will now show that all bipartite quantum states $\rho_{AB} \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$ with positive partial transpose are separable. This completely characterizes the entangled quantum states of two qubits. Using the cones from the last lecture, we formulate this result as:

Theorem 2.1. *We have* $\mathrm{Sep}\left(\mathbb{C}^2, \mathbb{C}^2\right) = PPT\left(\mathbb{C}^2, \mathbb{C}^2\right) \cap B(\mathbb{C}^2 \otimes \mathbb{C}^2)^+$.

In the following, we will go through the most elegant proof that I know. It is due to Aubrun and Szarek, and uses Brouwer's fixed point theorem at a key step:

Theorem 2.2 (Brouwer's fixed point theorem)**.** *Let $K \subset \mathcal{V}$ denote a convex and compact subset of a real Euclidean space $\mathcal{V}$. Any continuous function $f : K \to K$ has a fixed point, i.e., there exists $x \in K$ such that $f(x) = x$.*

To prove Theorem 2.1 we will show that every positive map $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ is decomposable, and thereby we will show that
$$B(\mathbb{C}^2 \otimes \mathbb{C}^2)^+ \vee \mathrm{PPT}\left(\mathbb{C}^2, \mathbb{C}^2\right) = \mathrm{BP}\left(\mathbb{C}^2, \mathbb{C}^2\right).$$

Taking duals of this identity finishes the proof. In the exercises, we have studied the unital[1] and trace-preserving positive maps $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$. We have shown the following facts:

[1] Such that $P(\mathbb{1}_{\mathcal{H}_A}) = \mathbb{1}_{\mathcal{H}_B}$

- For any unital an trace-preserving positive map $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ there exist unitaries $U, V \in \mathcal{U}\left(\mathbb{C}^2\right)$ such that

$$P_{U,V} = \mathrm{Ad}_U \circ P \circ \mathrm{Ad}_V$$

  is diagonal in the Pauli basis, i.e., such that there are $\lambda_i \in \mathbb{R}$ satisfying

$$P_{U,V}\left(\sigma_i\right) = \lambda_i \sigma_i,$$

  for any $i \in \{0,1,2,3\}$ with

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  Note that $\lambda_0 = 1$ by unitality.

- The map $P_{U,V}$ is positive if and only if $|\lambda_i| \leqslant 1$ for any $i \in \{1,2,3\}$.

- The map $P_{U,V}$ is completely positive if and only if

$$\begin{aligned} 1 + \lambda_1 + \lambda_2 + \lambda_3 &\geqslant 0 \\ 1 + \lambda_1 - \lambda_2 - \lambda_3 &\geqslant 0 \\ 1 - \lambda_1 + \lambda_2 - \lambda_3 &\geqslant 0 \\ 1 - \lambda_1 - \lambda_2 + \lambda_3 &\geqslant 0. \end{aligned}$$

  These inequalities define a tetrahedron Tetr_{CP} inside the cube $[-1,1]^3 \subset \mathbb{R}^3$.

- The map $P_{U,V}$ is completely copositive if and only if

$$\begin{aligned} 1 + \lambda_1 - \lambda_2 + \lambda_3 &\geqslant 0 \\ 1 + \lambda_1 + \lambda_2 - \lambda_3 &\geqslant 0 \\ 1 - \lambda_1 - \lambda_2 - \lambda_3 &\geqslant 0 \\ 1 - \lambda_1 + \lambda_2 + \lambda_3 &\geqslant 0. \end{aligned}$$

  These inequalities define another tetrahedron Tetr_{coCP} inside the cube $[-1,1]^3 \subset \mathbb{R}^3$.

- The intersection $\mathrm{Tetr}_{CP} \cap \mathrm{Tetr}_{coCP}$ is a regular octahedron (i.e., the dual of the cube $[-1,1]^3$).

- The union $\mathrm{Tetr}_{CP} \cup \mathrm{Tetr}_{coCP}$ is the *stellated octangula* and its convex hull is the entire cube $[-1,1]^3 \subset \mathbb{R}^3$. See Figure 2 for a picture of this geometric configuration.

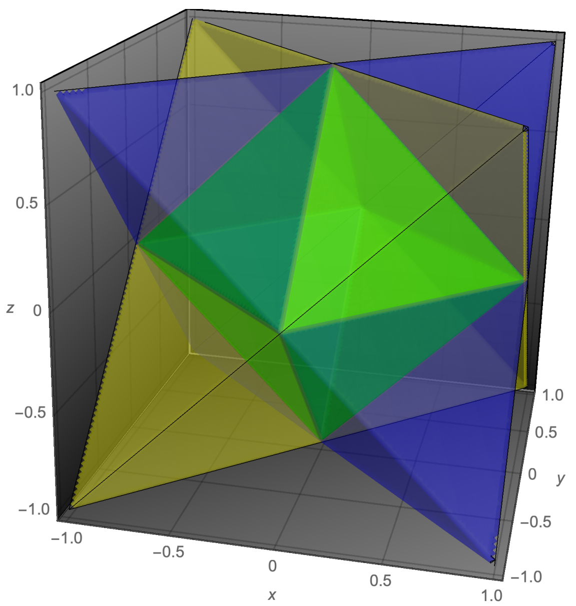

Figure 2: The tetrahedra Tetr_{CP} and Tetr_{coCP} (which one is which?) inside the cube $[-1,1]^3$

The last point implies the following lemma:

Lemma 2.3. *Any unital and trace-preserving positive map $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ is decomposable.*

The following normal form will allow us to reduce the general case to the previous lemma:

Theorem 2.4 (Sinkhorn normal form)**.** *Let $\mathcal{H}$ denote a complex Euclidean space. Consider a positive map $P : B(\mathcal{H}) \to B(\mathcal{H})$ such that $P(|\psi\rangle\langle\psi|) \in B(\mathcal{H})^+$ is invertible for any $|\psi\rangle \in \mathcal{H}$. Then, there exist invertible operators $X \in B(\mathcal{H})$ and $Y \in B(\mathcal{H})$ such that the linear map*

$$\tilde{P} = \mathrm{Ad}_Y \circ P \circ \mathrm{Ad}_X$$

is trace-preserving and unital, i.e., satisfies $\tilde{P}(\mathbb{1}_{\mathcal{H}}) = \mathbb{1}_{\mathcal{H}}$.

Note that the condition that $P(|\psi\rangle\langle\psi|) \in B(\mathcal{H})^+$ is invertible for any $|\psi\rangle \in \mathcal{H}$ is equivalent to saying that C_P belongs to the interior of $\mathrm{BP}\left(\mathbb{C}^2, \mathbb{C}^2\right)$.

Proof. Note that a linear map $L : B(\mathcal{H}) \to B(\mathcal{H})$ is trace-preserving if and only if its adjoint L^* is unital since

$$\mathrm{Tr}\left[L(X)\right] = \mathrm{Tr}\left[L^*\left(\mathbb{1}_{\mathcal{H}}\right) X\right],$$

which coincides with $\mathrm{Tr}\left[X\right]$ for every $X \in B(\mathcal{H})$ if and only if $L^*\left(\mathbb{1}_{\mathcal{H}}\right) = \mathbb{1}_{\mathcal{H}}$. To prove the Sinkhorn normal form consider the (non-linear!) inverse maps $\mathrm{inv} : \mathrm{int}\left(B(\mathcal{H})^+\right) \to \mathrm{int}\left(B(\mathcal{H})^+\right)$ mapping any strictly positive operator to its inverse. Consider now the non-linear map $\Lambda : D(\mathcal{H}) \to D(\mathcal{H})$ given by

$$\Lambda(X) = \frac{\mathrm{inv} \circ P^* \circ \mathrm{inv} \circ P(X)}{\mathrm{Tr}\left[\mathrm{inv} \circ P^* \circ \mathrm{inv} \circ P(X)\right]},$$

where we used that $P(\tau) \in \mathrm{int}\left(B(\mathcal{H})^+\right)$ for any $\tau \in D(\mathcal{H})$. The map Λ is a continuous map from the convex body $D(\mathcal{H})$ into itself. By Brouwer's fixed point theorem such a map has a fixed point $\rho \in D(\mathcal{H})$, i.e., a point satisfying $\Lambda(\rho) = \rho$. Defining $\sigma = P(\rho)$ this implies that

$$P^*(\sigma^{-1}) = \lambda\rho^{-1}, \tag{1}$$

for some $\lambda > 0$. Now set $X = \rho^{1/2}$ and $Y = \sigma^{-1/2}$ and

$$\tilde{P} = \mathrm{Ad}_Y \circ P \circ \mathrm{Ad}_X .$$

We can verify that

$$\tilde{P}\left(\mathbb{1}_{\mathcal{H}}\right) = \mathrm{Ad}_Y \circ P(\rho) = \sigma^{-1/2}\sigma\sigma^{-1/2} = \mathbb{1}_{\mathcal{H}},$$

and, using (1), we can verify that

$$\tilde{P}^*\left(\mathbb{1}_{\mathcal{H}}\right) = \left(\mathrm{Ad}_X \circ P^* \circ \mathrm{Ad}_Y\right)\left(\mathbb{1}_{\mathcal{H}}\right) = \rho^{1/2} P^*\left(\sigma^{-1}\right)\rho^{1/2} = \lambda \mathbb{1}_{\mathcal{H}}.$$

Finally, we note that

$$\dim(\mathcal{H}) = \mathrm{Tr}\left[\tilde{P}\left(\mathbb{1}_{\mathcal{H}}\right)\right] = \mathrm{Tr}\left[\tilde{P}^*\left(\mathbb{1}_{\mathcal{H}}\right)\right] = \lambda \dim\left(\mathcal{H}\right),$$

and we conclude that $\lambda = 1$. □

Now, we can prove Theorem 2.1:

Proof of Theorem 2.1. By Lemma 1.1 it suffices to prove that

$$B(\mathbb{C}^2 \otimes \mathbb{C}^2)^+ \vee \mathrm{PPT}\left(\mathbb{C}^2, \mathbb{C}^2\right) = \mathrm{BP}\left(\mathbb{C}^2, \mathbb{C}^2\right).$$

One inclusion is clear. For the other conclusion consider a point $C_P \in \mathrm{BP}\left(\mathbb{C}^2, \mathbb{C}^2\right)$ corresponding to a positive map $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ via the Choi-Jamiolkowski isomorphism. For $\epsilon > 0$ consider the map $P_\epsilon : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ given by

$$P_\epsilon(X) = P(X) + \epsilon \mathbb{1}_{\mathbb{C}^2} \mathrm{Tr}\left[X\right],$$

and note that $P_\epsilon(|\psi\rangle\langle\psi|) \in B(\mathcal{H})^+$ is invertible for any $|\psi\rangle \in \mathcal{H}$. For any $\epsilon > 0$ we conclude by Sinkhorn's normal form that there are invertible operators $A \in B(\mathbb{C}^2)$ and $B \in B(\mathbb{C}^2)$ (depending on ϵ) such that

$$\tilde{P}_\epsilon = \mathrm{Ad}_A \circ P_\epsilon \circ \mathrm{Ad}_B$$

is positive, unital and trace-preserving. By the exercises, we know that any such maps is decomposable, i.e., can be written as

$$\tilde{P}_\epsilon = T_1 + \vartheta \circ T_2$$

for completely positive maps $T_1, T_2 : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$. Now, we note that

$$\begin{aligned} P_\epsilon &= \mathrm{Ad}_{A^{-1}} \circ T_1 \circ \mathrm{Ad}_{B^{-1}} + \mathrm{Ad}_{A^{-1}} \circ \vartheta \circ T_2 \circ \mathrm{Ad}_{B^{-1}} \\ &= \mathrm{Ad}_{A^{-1}} \circ T_1 \circ \mathrm{Ad}_{B^{-1}} + \vartheta \circ \mathrm{Ad}_{(A^{-1})^T} \circ T_2 \circ \mathrm{Ad}_{B^{-1}}, \end{aligned}$$

is decomposable as well. Therefore, we have show that

$$C_{P_\epsilon} = C_P + \epsilon \mathbb{1}_{\mathbb{C}^2} \otimes \mathbb{1}_{\mathbb{C}^2} \in B(\mathbb{C}^2 \otimes \mathbb{C}^2)^+ \vee \mathrm{PPT}\left(\mathbb{C}^2, \mathbb{C}^2\right),$$

for any $\epsilon > 0$. Since $C_P \in \mathrm{BP}\left(\mathbb{C}^2, \mathbb{C}^2\right)$ was chosen freely, and since the cone $B(\mathbb{C}^2 \otimes \mathbb{C}^2)^+ \vee \mathrm{PPT}\left(\mathbb{C}^2, \mathbb{C}^2\right)$ is closed, we can conclude that

$$\mathrm{BP}\left(\mathbb{C}^2, \mathbb{C}^2\right) \subseteq B(\mathbb{C}^2 \otimes \mathbb{C}^2)^+ \vee \mathrm{PPT}\left(\mathbb{C}^2, \mathbb{C}^2\right).$$

This finishes the proof. □

3 Examples of entangled PPT operators

Consider complex Euclidean spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with dimensions d_A and d_B, respectively. If $(d_A, d_B) \notin \{(2,2),(2,3),(3,2)\}$, then we want to construct an entangled PPT operator (which we can always normalize to a quantum state). By Lemma 1.1 this can be achieved by constructing a point

$$W_{AB} \in \mathrm{BP}\left(\mathcal{H}_A, \mathcal{H}_B\right) \setminus \left(B(\mathcal{H}_A \otimes \mathcal{H}_B)^+ \vee \mathrm{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right)\right).$$

The existence of a quantum state $\rho_{AB} \in \mathrm{PPT}\left(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}\right) \cap B(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})^+$ such that

$$\mathrm{Tr}\left[\rho_{AB} W_{AB}\right] < 0,$$

follows from the hyperplane separation theorem[2]. To make our life a bit simpler, it is helpful to note the following facts:

- It is easy to check the inclusion $\rho_{AB} \in \mathrm{PPT}\left(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}\right) \cap B(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})^+$ for an operator ρ_{AB} by computing eigenvalues.
- If we come up with a pair $W_{AB} \in \mathrm{BP}\left(\mathcal{H}_A, \mathcal{H}_B\right)$ and $\rho_{AB} \in \mathrm{PPT}\left(\mathbb{C}^{d_A}, \mathbb{C}^{d_B}\right) \cap B(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})^+$ such that $\mathrm{Tr}\left[W_{AB}\rho_{AB}\right] < 0$, then we conclude that both ρ_{AB} is entangled (witnessed by W_{AB}) and that $W_{AB} \notin B(\mathcal{H}_A \otimes \mathcal{H}_B)^+ \vee \mathrm{PPT}\left(\mathcal{H}_A, \mathcal{H}_B\right)$ (by Lemma 1.1).

The most simple construction of such a pair is based on the following notion:

Definition 3.1 (Unextendible product set)**.** *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ denote complex Euclidean spaces. A finite set of product vectors*

$$\mathcal{S} = \{|a_1\rangle \otimes |b_1\rangle, \ldots, |a_m\rangle \otimes |b_m\rangle\} \subset \mathcal{H}_A \otimes \mathcal{H}_B$$

is called an unextendible product set (UPS) *if the following conditions hold:*

1. *The vectors in $\mathcal{S}$ are orthonormal.*
2. *We have $m < \dim\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$.*
3. *The orthogonal complement $\mathcal{S}^\perp$ does not contain any non-zero product vector $|x\rangle \otimes |y\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.*

Let us construct an example of a UPS:

Example 1 (Unextendible product bases for $\mathbb{C}^3 \otimes \mathbb{C}^3$)**.** Consider the normalized vectors

$$\begin{aligned}
|v_1\rangle &= |1\rangle, \\
|v_2\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right), \\
|v_3\rangle &= \frac{1}{\sqrt{2}}\left(|2\rangle - |3\rangle\right), \\
|v_4\rangle &= |3\rangle, \\
|v_5\rangle &= \frac{1}{\sqrt{3}}\left(|1\rangle + |2\rangle + |3\rangle\right).
\end{aligned}$$

These vectors have two important properties that can be easily verified:

- Any triple $\{|v_i\rangle, |v_j\rangle, |v_k\rangle\}$ of distinct vectors spans $\mathbb{C}^3$.

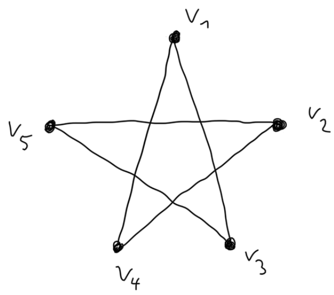

Figure 3: Orthogonality relations as edges.

- The orthogonality relations between the vectors can be visualized by the graph in Figure 3 with vertices $v_1, \ldots, v_5$ and where we draw an edge between vertices v_i and v_j if and only if $\langle v_i | v_j \rangle = 0$. Note that these edges are the diagonals of a regular pentagon.

To construct an UPS, let us first consider the set

$$\mathcal{S}_\pi = \{|v_i\rangle \otimes |v_{\pi(i)}\rangle \ : \ i \in \{1, \ldots, 5\}\},$$

for some permutation $\pi \in S_5$. We claim that the orthogonal complement $\mathcal{S}_\pi^\perp$ for any permutation $\pi \in S_5$ does not contain any product vectors. To see this, assume that $|x\rangle \otimes |y\rangle \in \mathcal{S}_\pi^\perp$, i.e.,

$$\langle x|v_i\rangle \langle y|v_{\pi(i)}\rangle = 0 \quad \text{for every } i \in \{1, \ldots, 5\}.$$

By the pidgeonhole principle there are three distinct indices $i, j, k \in \{1, \ldots, 5\}$ such that

$$\langle x|v_i\rangle = \langle x|v_j\rangle = \langle x|v_k\rangle = 0 \quad \text{or} \quad \langle y|v_{\pi(i)}\rangle = \langle y|v_{\pi(j)}\rangle = \langle y|v_{\pi(k)}\rangle = 0.$$

Since the triples $\{|v_i\rangle, |v_j\rangle, |v_k\rangle\}$ and $\{|v_{\pi(i)}\rangle, |v_{\pi(j)}\rangle, |v_{\pi(k)}\rangle\}$ span $\mathbb{C}^3$, we conclude that either $|x\rangle = 0$ or $|y\rangle = 0$.

Now, we only have to find a permutation $\pi \in S_5$, such that the vectors in $\mathcal{S}_\pi$ are orthogonal. There is a very nifty graph-theoretic construction to do this: Start with the vertices $v_1, \ldots, v_5$ forming a pentagon. In the following, we will distinguish between diagonals and sides of the pentagon (see Figure 4).

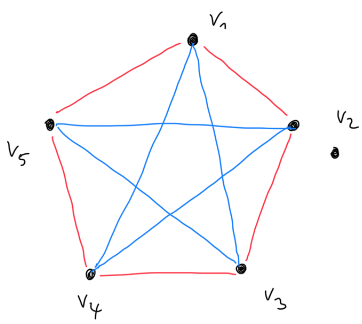

Figure 4: Diagonals in blue and sides in red.

We will now represent each product vector $|v_i\rangle \otimes |v_{\pi(i)}\rangle$ by a directed edge (arrow) from v_i to $v_{\pi(i)}$. See Figure 5 for two examples.

[2]Here, ρ_{AB} defines a hyperplane separating W_{AB} from the closed cone $B(\mathcal{H}_A \otimes \mathcal{H}_B)^+ \vee \mathrm{PPT}\,(\mathcal{H}_A, \mathcal{H}_B)$.

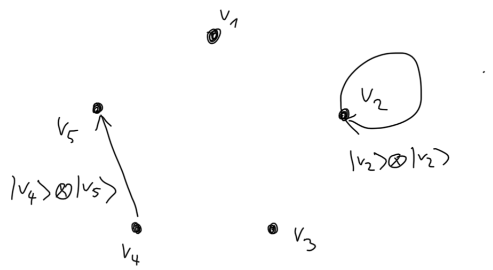

Figure 5: Some product vectors represented by arrows.

The set $\mathcal{S}_\pi$ for a permutation $\pi \in S_5$ corresponds to a graph with vertices $v_1, \ldots, v_5$ and directed edges such that each vertex v_i is the starting point of exactly one edge and the end point of exactly one edge. In particular, decomposing the vertex set into disjoint cycles (see Figure 6) would ensure this condition.

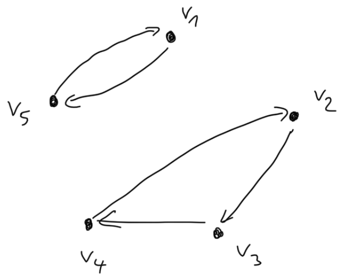

Figure 6: Decomposing vertices into disjoint cycles.

How can we enforce orthogonality between vectors in $\mathcal{S}_\pi$? Note that in this graph-theoretic consideration the orthogonality

$$|v_i\rangle \otimes |v_{\pi(i)}\rangle \perp |v_j\rangle \otimes |v_{\pi(j)}\rangle$$

is equivalent to the condition that a diagonal of the pentagon has to connect either the two starting points v_i and v_j or the two end points $v_{\pi(i)}$ and $v_{\pi(j)}$ of the arrows $v_i \to v_{\pi(i)}$ and $v_j \to v_{\pi(j)}$. See Figure 7 for an example of a pair of arrows not leading to orthogonal vectors.

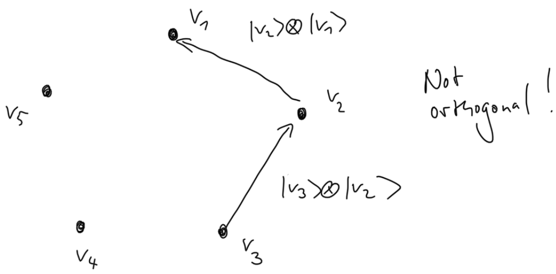

Figure 7: An example of two arrows not leading to orthogonal vectors.

Two examples of pairs of arrows leading to orthogonal vectors can be seen in Figure 8.

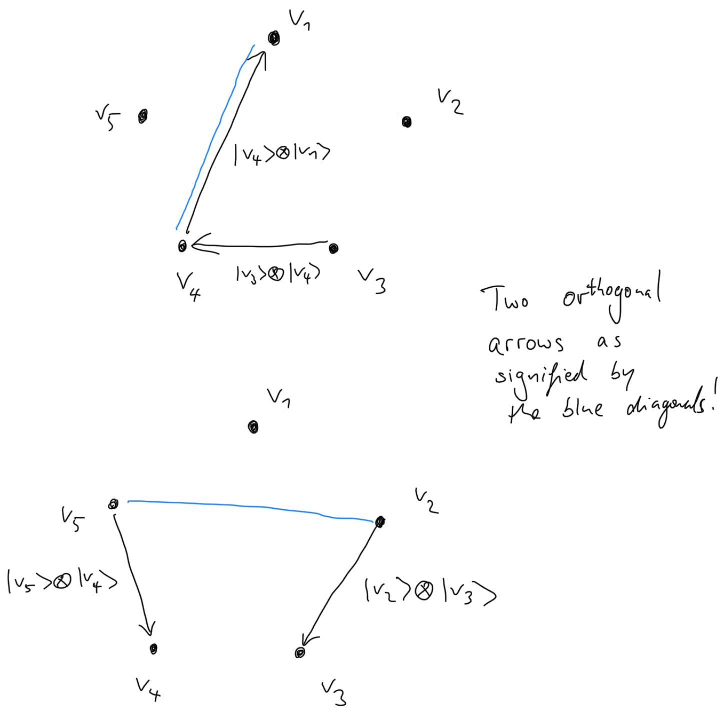

Figure 8: Two orthogonal pairs of arrows.

Note that disjoint arrows involving two vertices always lead to orthogonal vectors! So, how can we enforce orthogonality between vectors in $\mathcal{S}_\pi$? A simple way is to decompose the vertices into disjoint cycles with edges alternating between sides of the pentagon and diagonals of the pentagon and that at most one cycle has length one. This can be seen in Figure 9

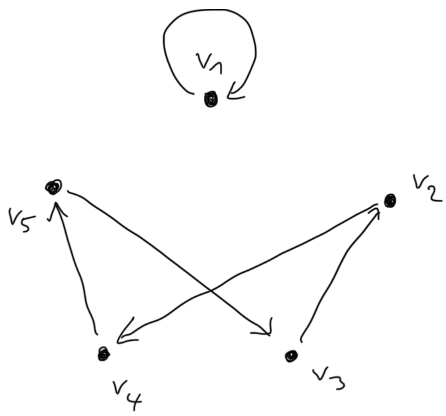

Figure 9: The solution.

We conclude that the set

$$\mathcal{S} = \{|v_1\rangle \otimes |v_1\rangle, |v_2\rangle \otimes |v_4\rangle, |v_3\rangle \otimes |v_2\rangle, |v_4\rangle \otimes |v_5\rangle, |v_5\rangle \otimes |v_3\rangle\},$$

is an UPS in $\mathbb{C}^3 \otimes \mathbb{C}^3$.

To construct an entangled PPT state in $\mathcal{D}\left(\mathbb{C}^3 \otimes \mathbb{C}^3\right)$ consider the UPS $\mathcal{S}$ constructed in the previous example. Let $P_\mathcal{S} \in \mathrm{Proj}\,(\mathcal{H}_A \otimes \mathcal{H}_B)$ denote the projector onto $\mathrm{span}\,(\mathcal{S})$ given by

$$P_{\mathcal{S}} = \sum_{i=1}^{m} |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|.$$

Since $\mathcal{S}$ is an UPS, we have

$$\epsilon = \inf\{(\langle x| \otimes \langle y|)\, P_{\mathcal{S}}\, (|x\rangle \otimes |y\rangle) \ : \ |x\rangle, |y\rangle \in \mathbb{C}^3, \langle x|x\rangle = \langle y|y\rangle = 1\} > 0. \tag{2}$$

Consider now the operator

$$W_{AB} = P_{\mathcal{S}} - \epsilon \mathbb{1}_{\mathcal{H}_A} \otimes \mathbb{1}_{\mathcal{H}_B}.$$

By (2) we have

$$(\langle x| \otimes \langle y|)\, W_{AB}\, (|x\rangle \otimes |y\rangle) \geqslant 0,$$

for all $|x\rangle, |y\rangle \in \mathbb{C}^3$, and we conclude that $W_{AB} \in \mathrm{BP}\left(\mathbb{C}^3 \otimes \mathbb{C}^3\right)$. Next, consider the normalized projector onto the orthogonal complement $\mathcal{S}^{\perp}$ given by

$$\rho_{AB} = \frac{\mathbb{1}_{\mathcal{H}_A} \otimes \mathbb{1}_{\mathcal{H}_B} - P_{\mathcal{S}}}{\mathrm{Tr}\left[\mathbb{1}_{\mathcal{H}_A} \otimes \mathbb{1}_{\mathcal{H}_B} - P_{\mathcal{S}}\right]}.$$

Clearly, we have $\rho_{AB} \in \mathcal{D}\left(\mathbb{C}^3 \otimes \mathbb{C}^3\right)$ and since the vectors in $\mathcal{S}$ have real entries, we have $\rho_{AB}^{\Gamma} = \rho_{AB}$. We conclude that

$$\rho_{AB} \in \mathrm{PPT}\left(\mathbb{C}^3, \mathbb{C}^3\right) \cap B(\mathbb{C}^3 \otimes \mathbb{C}^3)^{+}.$$

Finally, note that

$$\mathrm{Tr}\left[W_{AB}\rho_{AB}\right] = -\epsilon < 0,$$

and we conclude that ρ_{AB} is entangled, and $W_{AB} = C_Q$ for a non-decomposable positive map $Q : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$.