19th April, 2022

COURSE 4430— Quantum information theory

Mandatory assignment 1 of 1

Submission deadline

Thursday 28th April 2022, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Try to solve as many problems as possible. To pass the assigment answering correctly half of the exercises is sufficient. Here, an exercise means the problems 2 and 3, and each of the subproblems appearing in problems 1, 4, 5, and 6. There are 12 exercises in total.

Problem 1.

1. For $d \geq 2$ find a set of unitary operators $\{U_1, \dots, U_{d^2}\}$ such that

$$\mathrm{Tr}\left[X\right] \frac{\mathbb{1}_{\mathbb{C}^d}}{d} = \frac{1}{d^2} \sum_{i=1}^{d^2} \mathrm{Ad}_{U_i}(X),$$

   for each $X \in B(\mathbb{C}^d)$. Prove that your set satisfies the desired identity.

2. Find vectors $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle\} \in \mathbb{C}^2$ such that

$$|\langle \psi_i | \psi_j \rangle|^2 = \begin{cases} 1 & \text{if } i = j \\ \frac{1}{3} & \text{if } i \neq j \end{cases}.$$

Problem 2 (Channels with vanishing classical capacity)**.** Let $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ denote a quantum channel. Show that $C(T) = 0$ if and only if T is constant, i.e., there exists a quantum state $\sigma \in D(\mathcal{H}_B)$ such that $T(X) = \mathrm{Tr}\left[X\right] \sigma$.

Problem 3. Let $T : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ denote a unital quantum channel that is *not* of the form $T = \mathrm{Ad}_U$ for any unitary $U \in \mathcal{U}\left(\mathbb{C}^2\right)$. Show that there exists some $p > 0$, a unitary $V \in \mathcal{U}\left(\mathbb{C}^2\right)$ and a unital entanglement breaking channel $E : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ such that

$$T = (1 - p) \mathrm{Ad}_V + pE.$$

Hint: Stare at a tetrahedron!

Problem 4. Consider the linear map $S : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ given by

$$S(X) = \frac{2}{3} \left(\mathrm{Tr}\left[X\right] \mathbb{1}_2 - \frac{1}{2} X^T \right).$$

1. Show that S is a quantum channel.

2. Compute the regularized minimum output entropy

$$\lim_{n \to \infty} \frac{1}{n} H_{\min}\left(S^{\otimes n}\right).$$

Problem 5. Consider the linear map $S : B(\mathbb{C}^3) \to B(\mathbb{C}^3)$ given by

$$S(X) = \frac{1}{2}\left(\mathrm{Tr}\left[X\right]\mathbb{1}_3 - X^T\right).$$

1. Compute the maximal output p-norm

$$\nu_p(S) = \sup_{\rho \in D(\mathbb{C}^3)} \|S(\rho)\|_p.$$

2. Find a quantum state $\rho \in D\left(\mathbb{C}^3 \otimes \mathbb{C}^3\right)$ satisfying

$$\|(S \otimes S)(\rho)\|_p > \nu_p(S)^2,$$

whenever p is large enough.

Problem 6. Let $\rho_A \in D\left(\mathcal{H}_A\right)$ denote a quantum state with purification $|\psi_{AB}\rangle\langle\psi_{AB}| \in \mathrm{Proj}\left(\mathcal{H}_A \otimes \mathcal{H}_B\right)$, and consider a decomposition $\rho_A = \sum_{n=1}^{N} p_n \rho_n$ with quantum states $\rho_n \in D(\mathcal{H}_A)$ and a probability distribution $p \in \mathcal{P}\left(\{1, \ldots, N\}\right)$. We aim to construct an instrument $\{T_n\}_{n=1}^{N}$ with $T_n : B(\mathcal{H}_B) \to B(\mathcal{H}_B)$ such that

$$\mathrm{Tr}_B\left[\left(\mathrm{id}_A \otimes T_n\right)\left(|\psi_{AB}\rangle\langle\psi_{AB}|\right)\right] = p_n \rho_n,$$

for any $n \in \{1, \ldots, N\}$. Follow the steps below:

1. Argue that without loss of generality we can assume $|\psi_{AB}\rangle = \mathrm{vec}\left(\sqrt{\rho_A}^T\right)$.

2. Show that

$$\mathrm{Tr}_B\left[\left(\mathrm{id}_A \otimes L\right)\left(\mathrm{vec}\left(\sqrt{\rho}^T\right)\mathrm{vec}\left(\sqrt{\rho}^T\right)^\dagger\right)\right] = \sqrt{\rho} L^*\left(\mathbb{1}_{H_B}\right)^T \sqrt{\rho},$$

for any completely positive map $L : B(H_B) \to B(H_B)$ and any $\rho \in D(\mathcal{H}_A)$.

3. Find operators $K_n \in B(\mathcal{H}_B)$ for every $n \in \{1, \ldots, N\}$ such that

$$\sqrt{\rho_A}(K_n^\dagger K_n)^T \sqrt{\rho_A} = p_n \rho_n,$$

and such that $\sum_{n=1}^{N} K_n^\dagger K_n = \mathbb{1}_{\mathcal{H}_B}$.

Hint: Consider first the case where ρ_A is invertible and use the Moore-Penrose pseudoinverse for the general case. Note that we need to ensure that $\sum_{n=1}^{N} K_n^\dagger K_n = \mathbb{1}_{\mathcal{H}_B}$.

4. Construct an instrument $\{T_n\}_{n=1}^{N}$ with $T_n : B(\mathcal{H}_B) \to B(\mathcal{H}_B)$ such that

$$\mathrm{Tr}_B \left[(\mathrm{id}_A \otimes T_n) \left(|\psi_{AB}\rangle\langle\psi_{AB}| \right) \right] = p_n \rho_n,$$

for any $n \in \{1, \ldots, N\}$.