EXERCISES 12

Excercise 1 (Norms of block matrices)**.** Let $\mathcal{H}$ denote a complex Euclidean space. We will show that

$$\| \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes X_{i,j} \|_\infty \leqslant d \max_{i,j} \|X_{i,j}\|_\infty .$$

for any operators $X_{i,j} \in B(\mathcal{H})$ for $i,j \in \{1,\dots,d\}$. Follow the steps below:

(1) Recall that $\|X\|_\infty = \max_{v,w} |\langle v|X|w\rangle|$ where the maximum is over vectors $|v\rangle, |w\rangle \in \mathcal{H}$ satisfying $\langle v|v\rangle = \langle w|w\rangle = 1$. Use this to show that

$$\| \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes X_{i,j} \|_\infty \leqslant \| \sum_{i,j=1}^{d} \|X_{i,j}\|_\infty |i\rangle\langle j| \|_\infty .$$

Hint: Note that vectors $|v\rangle \in \mathbb{C}^d \otimes \mathcal{H}$ can be written as $|v\rangle = \sum_{i=1}^{d} |i\rangle \otimes |v_i\rangle$ with $|v_i\rangle \in \mathcal{H}$ for each i.

(2) Use equivalence of the Schatten-p-norms to show that

$$\| \sum_{i,j=1}^{d} \|X_{i,j}\|_\infty |i\rangle\langle j| \|_\infty \leqslant \left(\sum_{i,j=1}^{d} \|X_{i,j}\|_\infty^2 \right)^{1/2} .$$

(3) Finally, show that

$$\left(\sum_{i,j=1}^{d} \|X_{i,j}\|_\infty^2 \right)^{1/2} \leqslant d \max_{i,j} \|X_{i,j}\|_\infty .$$

Excercise 2 (The diamond norm)**.** For a linear map $L : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ we define

$$\|L\|_\diamond = \sup_{n\in\mathbb{N}} \|\mathrm{id}_n \otimes L\|_{1\to 1} .$$

(1) Use the Schmidt decomposition to show that

$$\|\mathrm{id}_n \otimes L\|_{1\to 1} \leqslant \|\mathrm{id}_{B(\mathcal{H}_A)} \otimes L\|_{1\to 1},$$

and conclude that $\|L\|_\diamond < \infty$. **Hint:** Remember the extreme points of the $\|\cdot\|_1$-unit ball.

(2) Use duality of norms and Exercise 1 to show that

$$\|L\|_\diamond \leqslant \dim(\mathcal{H}_A)\, \|L\|_{1\to 1} .$$

(3) Consider the transpose map $\vartheta_d : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ in the computational basis and show that

$$\|\vartheta_d\|_\diamond = d.$$

Note that this shows that the general upper bound on the diamond norm is sharp.

(4) Show that we have

$$\|L_1 \otimes L_2\|_\diamond \leqslant \|L_1\|_\diamond \|L_2\|_\diamond ,$$

for any pair of linear maps $L_1 : B(\mathcal{H}_{A_1}) \to B(\mathcal{H}_{B_1})$ and $L_2 : B(\mathcal{H}_{A_2}) \to B(\mathcal{H}_{B_2})$.

Excercise 3 (Bose-symmetry)**.** Let $\mathcal{H}$ be a complex Euclidean space and $N \in \mathbb{N}$. We will now prove two important facts about the Bose-symmetric operators and their relationship with the symmetric subspace of operators.

(1) Show that
$$B\left(\mathcal{H}^{\vee N}\right) = \mathrm{span}_{\mathbb{C}}\{|v\rangle\langle v|^{\otimes N}\},$$
and that
$$B\left(\mathcal{H}^{\vee N}\right)_{sa} = \mathrm{span}_{\mathbb{R}}\{|v\rangle\langle v|^{\otimes N}\}$$
Hint: Define $|v_{\alpha,\beta}\rangle = e^{i\alpha}|x\rangle + e^{i\beta}|y\rangle$ and find a function $f(\alpha,\beta)$ satisfying
$$|x\rangle\langle y|^{\otimes N} = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(\alpha,\beta)|v_{\alpha,\beta}\rangle\langle v_{\alpha,\beta}|^{\otimes n} d\alpha d\beta.$$

(2) Show that any quantum state $\rho \in D\left(\mathcal{H}^{\otimes N}\right) \cap B(\mathcal{H})^{\vee N}$ admits a purification $|\psi\rangle \in (\mathcal{H}\otimes\mathcal{H})^{\vee N}$. **Hint:** Use $|\psi\rangle = \mathrm{vec}\left(\sqrt{\rho}\right)$ with an appropriate ordering of the tensor factors.

Excercise 4 (The de-Finetti theorem and its consequences)**.** For $m \in \mathbb{N}$ consider some pure state $|\psi\rangle \in \mathcal{H}^{\vee m}$. We will show that for any $n \in \mathbb{N}$ satisfying $n \leqslant m$, there exists a quantum state
$$\sigma \in \mathrm{conv}\left(|v\rangle\langle v|^{\otimes n} \ : \ |v\rangle \in \mathcal{H}, \langle v|v\rangle = 1\right),$$
such that
$$\| \mathrm{Tr}_{m\to n}\left(|\psi\rangle\langle\psi|\right) - \sigma\|_1 \leqslant \frac{n(d+n)}{m+d},$$
where $d = \dim(\mathcal{H})$ and $\mathrm{Tr}_{m\to n}$ denotes the partial trace over any $(m-n)$ of the m tensor factors. In the following, we use
$$d[k] = \binom{k+d-1}{k}.$$
Follow the steps below:

(1) Define the linear map $\mathrm{clone}_{n\to m} : B\left(\mathcal{H}^{\vee n}\right) \to B\left(\mathcal{H}^{\vee m}\right)$ by
$$\mathrm{clone}_{n\to m}(X) = \frac{d[n]}{d[m]} P^m_{sym}\left(X \otimes \mathbb{1}_{\mathcal{H}}^{\otimes(m-n)}\right) P^m_{sym},$$
and the linear map $\mathrm{MP}_{m\to n} : B(\mathcal{H}^{\vee m}) \to B(\mathcal{H}^{\vee n})$ by
$$\mathrm{MP}_{m\to n}\left(X\right) = d[m]\int_{\mathcal{U}(\mathcal{H})} \langle\phi_U^{\otimes m}|X|\phi_U^{\otimes m}\rangle |\phi_U\rangle\langle\phi_U|^{\otimes n} d\eta(U),$$
where $|\phi_U\rangle = U|0\rangle$ for any $U \in \mathcal{U}\left(\mathcal{H}\right)$. Verify that these maps are quantum channels.

(2) Show that
$$\langle |b\rangle\langle b|^{\otimes n}, \mathrm{MP}_{m\to n}\left(|a\rangle\langle a|^{\otimes m}\right)\rangle_{HS} = \frac{d[m]}{d[m+n]}\sum_{s=0}^{n} \frac{\binom{m}{s}\binom{n}{s}}{\binom{n+m}{s}} |\langle a|b\rangle|^{2s},$$
for $|a\rangle, |b\rangle \in \mathcal{H}$ satisfying $\langle a|a\rangle = \langle b|b\rangle = 1$. **Hint:** Find a symmetric projection P^{n+m}_{sym} and write is as a sum involving the unitaries U_σ for $\sigma \in S_{n+m}$.

(3) Use the prvious subexercise to show the identity
$$\mathrm{MP}_{m\to n} = \frac{d[m]}{d[m+n]}\sum_{s=0}^{n} \frac{d[n]}{d[s]} \frac{\binom{m}{s}\binom{n}{s}}{\binom{m+n}{s}} \,\mathrm{clone}_{s\to n} \circ \mathrm{Tr}_{m\to s}\,.$$

(4) By splitting off the summand for $s = n$ show that

$$\mathrm{MP}_{m\to n} = (1-\epsilon_{m,n,d})\operatorname{Tr}_{m\to n} + \epsilon_{m,n,d} R,$$

for some quantum channel $R : B(\mathcal{H}^{\vee m}) \to B(\mathcal{H}^{\vee n})$ and some

$$1-\epsilon_{m,n,d} = \frac{d[m]}{d[n+m]} \frac{\binom{m}{n}}{\binom{m+n}{n}} \geqslant 1 - \frac{n(d+n)}{m+d}.$$

(5) Compare the norm of $\mathrm{MP}_{m\to n}(|\psi\rangle\langle\psi|)$ and $\operatorname{Tr}_{m\to n}(|\psi\rangle\langle\psi|)$ to show the quantum de-Finetti theorem.