EXERCISES 14

Excercise 1 (The data processing inequality)**.** Show that

$$I_c(\sigma, T) = D\left((\mathrm{id}_{A'} \otimes T)\left(|\psi_{A'A}\rangle\langle\psi_{A'A}|\right) \| \psi_{A'} \otimes T(\sigma)\right) - H(\sigma),$$

for any quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and any quantum state $\sigma \in D(\mathcal{H}_A)$ with purification

$$|\psi_{A'A}\rangle = \mathrm{vec}\left(\sqrt{\sigma}\right).$$

Use this representation to show the data-processing inequality

$$I_c(\sigma, D \circ T) \leqslant I_c(\sigma, T),$$

for any quantum channel $D : B(\mathcal{H}_B) \to B(\mathcal{H}_C)$.

Excercise 2 (Additivity of the coherent information for degradable channels)**.** A quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is called *degradable* if there exists a quantum channel $R : B(\mathcal{H}_B) \to B(\mathcal{H}_E)$ such that

$$T^c = R \circ T,$$

for some complementary channel $T^c : B(\mathcal{H}_A) \to B(\mathcal{H}_E)$. Show that

$$I_c(T^{\otimes k}) = k I_c(T),$$

for any degradable quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ and any $k \in \mathbb{N}$.
Hint: Consider degradable quantum channels $T_1 : B(\mathcal{H}_{A_1}) \to B(\mathcal{H}_{B_1})$ and $T_2 : B(\mathcal{H}_{A_2}) \to B(\mathcal{H}_{B_2})$, and play around with the relative entropy

$$D((T_1^c \otimes T_2^c)(\sigma_{A_1A_2}) \| T_1^c(\sigma_{A_1}) \otimes T_2^c(\sigma_{A_2})),$$

for some $\sigma_{A_1A_2} \in D(\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2})$, and where $\sigma_{A_1} = \mathrm{Tr}_{A_2}[\sigma_{A_1A_2}]$ and $\sigma_{A_2} = \mathrm{Tr}_{A_1}[\sigma_{A_1A_2}]$. You might want to use the data-processing inequality.

Excercise 3 (Quantum capacity of the erasure channel)**.** Consider the erasure channel $E_\lambda : B(\mathbb{C}^d) \to B(\mathbb{C}^{d+1})$ for $\lambda \in [0, 1]$ given by

$$E_\lambda(X) = (1 - \lambda)X \oplus 0 + \lambda \,\mathrm{Tr}[X]\, |d + 1\rangle\langle d + 1|.$$

Show that

$$Q(E_\lambda) = (1 - 2\lambda)\log(d).$$

Hint: Use the previous exercise.