19th April, 2023

COURSE 4430— Quantum information theory

Mandatory assignment 1 of 1

Submission deadline

Thursday 4th May 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Alice and Bob need your help! Two researchers Alice and Bob share many copies of some entangled quantum state. They would like to transform these copies into a single maximally entangled state, that they can use for various tasks. Due to experimental restrictions they are only allowed to apply quantum channels and measurements locally to their respective quantum systems, but they may communicate classical measurement outcomes to each other. When are they able to succeed in this task? The goal of this assignment is to understand a particular case where this is possible. For simplicity, we will allow Alice and Bob to apply any quantum channel with Kraus operators of the form $K_A \otimes K_B$, where K_A acts on Alice's systems and K_B acts on Bob's systems. This class of quantum channels is called *separable quantum channels.*

Try to solve as many problems as possible. To pass the assigment answering correctly half of the 11 subproblems is sufficient.

Problem 1. Consider $\tau_\alpha \in B(\mathbb{C}^d \otimes \mathbb{C}^d)$ of the form

$$\tau_\alpha = \alpha \frac{\mathbb{1}_d \otimes \mathbb{1}_d}{d^2} + (1-\alpha)\omega_d,$$

where $\omega_d \in D\left(\mathbb{C}^d \otimes \mathbb{C}^d\right)$ denotes the (normalized!) maximally entangled quantum state.

1. Determine for which parameters $\alpha \in \mathbb{R}$ the operator τ_α determines a quantum state and for which $\alpha \in \mathbb{R}$ it is PPT.

2. Is there any $\alpha \in \mathbb{R}$ for which τ_α is PPT and entangled? Give an argument for your answer.

We will refer to the quantum states of the above form as *nice states.*

Problem 2. Consider the linear map $R : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ given by

$$R(X) = \mathrm{Tr}\,[X]\,\mathbb{1}_d - X,$$

for any $X \in B(\mathbb{C}^d)$.

1. Show that R is positive, but not completely positive.

2. Show that $R = \vartheta_d \circ S$ for a completely positive map $S : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ and the transpose map ϑ_d in some basis. Show that S is a unitary quantum channel, when $d = 2$.

3. Assume that $\rho \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ is a quantum state such that

$$(\mathrm{id}_{B(\mathbb{C}^d)} \otimes R)(\rho) \notin B(\mathbb{C}^d \otimes \mathbb{C}^d)^+.$$

Show that there exists an operator $X \in B(\mathbb{C}^d)$ such that

$$\mathrm{Tr}\left[\sigma\omega_d\right] > \frac{1}{d},$$

where ω_d denotes the maximally entangled state, and

$$\sigma = \frac{(X^\dagger \otimes \mathbb{1}_d)\rho(X \otimes \mathbb{1}_d)}{\mathrm{Tr}\left[(X^\dagger \otimes \mathbb{1}_d)\rho(X \otimes \mathbb{1}_d)\right]}.$$

Describe a separable instrument that can transform ρ to σ with some non-zero probability.

Problem 3. For $d \in \mathbb{N}$ let $\tau_\alpha \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ denote the nice states from the first exercise.

1. Show that

$$(U \otimes \bar{U})\tau_\alpha = \tau_\alpha(U \otimes \bar{U}),$$

for every unitary $U \in \mathcal{U}\left(\mathbb{C}^d\right)$ and any $\alpha \in \mathbb{R}$.

2. Consider a quantum state $\sigma \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ shared between two parties and assume that

$$\mathrm{Tr}\left[\sigma\omega_d\right] > \frac{1}{d}$$

Show that there exists a separable quantum channel $\Lambda : B(\mathbb{C}^d \otimes \mathbb{C}^d) \to B(\mathbb{C}^d \otimes \mathbb{C}^d)$ such that

$$\Lambda(\sigma) = \tau_\alpha$$

is some entangled nice state. Find a formula in terms of $\mathrm{Tr}\left[\sigma\omega_d\right]$ for the parameter α that you can achieve using the separable quantum channel Λ of your choice.

Problem 4. Consider the quantum channel $\Delta : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ given by

$$\Delta(X) = \sum_{k=1}^{d} \langle k|X|k\rangle |k\rangle\langle k|,$$

and the linear map $C : B(\mathbb{C}^d \otimes \mathbb{C}^d) \to B(\mathbb{C}^d \otimes \mathbb{C}^d)$ given by

$$C(Y) = \sum_{k,l=1}^{d} \left(\langle k| \otimes \langle k|\right)Y\left(|l\rangle \otimes |l\rangle\right)|k\rangle\langle l| \otimes |k\rangle\langle l|.$$

Furthermore, define the operator $W : \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$ given by

$$W = \sum_{k,l=0}^{d-1} |k\rangle\langle k| \otimes |k+l\rangle\langle l|,$$

where addition is modulo d.

1. Show that for every $n \in \{0, 1, \ldots, d-1\}$ the map $E_n : B(\mathbb{C}^d \otimes \mathbb{C}^d) \to B(\mathbb{C}^d \otimes \mathbb{C}^d)$ given by

   $$E_n(X) = (\mathrm{id}_{A_1B_1} \otimes M_n^{A_2B_2}) \left[(W_{A_1A_2} \otimes W_{B_1B_2}) \left(X_{A_1B_1} \otimes \tau_\alpha^{A_2B_2} \right) (W_{A_1A_2} \otimes W_{B_1B_2})^\dagger \right],$$

   for

   $$M_n(X) = (\langle n| \otimes \langle n|) X (|n\rangle \otimes |n\rangle)$$

   is given by

   $$E_n = \frac{\alpha}{d^2} \Delta^{\otimes 2} + \frac{1-\alpha}{d} C.$$

   Here, τ_α is a nice state from the previous exercises. How would you describe to a physicist how to implement the transformation E_n with some non-zero probability provided that they have a nice state available?

2. Let us denote by $T_\alpha : B(\mathbb{C}^d \otimes \mathbb{C}^d) \to B(\mathbb{C}^d \otimes \mathbb{C}^d)$ the completely positive map

   $$T_\alpha = \frac{\alpha}{d^2} \Delta^{\otimes 2} + \frac{1-\alpha}{d} C,$$

   from the previous subexercise. Compute the fidelity

   $$F(\alpha) = \frac{\langle \Omega_d | T_\alpha(\rho_\alpha) | \Omega_d \rangle}{d \operatorname{Tr}\left[T_\alpha(\rho_\alpha) \right]},$$

   where $|\Omega_d\rangle$ is the (unnormalized) maximally entangled vector.

3. Find a separable quantum channel $\Lambda : B(\mathbb{C}^d \otimes \mathbb{C}^d) \to B(\mathbb{C}^d \otimes \mathbb{C}^d)$ such that

   $$\Lambda \left(\frac{T_\alpha(\rho_\alpha)}{\operatorname{Tr}\left[T_\alpha(\rho_\alpha) \right]} \right) = \rho_{a(\alpha)},$$

   with

   $$a(\alpha) = \frac{\alpha \left(2d - (d-1)\alpha \right)}{(d+1)\left((d-1)\alpha^2 - 2(d-1)\alpha + d \right)}.$$

   Show that for every α such that ρ_α is entangled, we have

   $$a^{(n)}(\alpha) \to 0,$$

   as $n \to \infty$.

4. Show that whenever ρ_α is entangled, then there exists a sequence of separable quantum channels $\Lambda_n : B(\mathbb{C}^{d^n} \otimes \mathbb{C}^{d^n}) \to B(\mathbb{C}^2 \otimes \mathbb{C}^2)$ such that
$$\|\Lambda_n(\rho_\alpha^{\otimes n}) - \omega_2\|_1 \to 0,$$
as $n \to \infty$. Using previous exercises, show that whenever $\rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is not PPT, then there exists a sequence $\Lambda'_n : B(\mathbb{C}^{d^n} \otimes \mathbb{C}^{d^n}) \to B(\mathbb{C}^2 \otimes \mathbb{C}^2)$ of separable quantum channels such that
$$\|\Lambda'_n(\rho^{\otimes n}) - \omega_2\|_1 \to 0,$$
as $n \to \infty$. Find a way to do the same for any $\rho \in D(\mathbb{C}^{d'} \otimes \mathbb{C}^2)$ that is not PPT and where $d' > 2$.