EXERCISES 1

1. Training

Excercise 1 (Classes of linear operators)**.**
Consider the Euclidean space $\mathcal{H} = \mathbb{C}^2$. Which of the following operators in $B(\mathcal{H})$ (written as matrices in the computational basis) are normal, selfadjoint, positive, projections, and/or unitaries:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad D = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 3 \\ 3 & -2 \end{pmatrix} \quad F = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \quad G = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad J = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & i \\ i & 2 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Excercise 2 (The Kronecker product)**.**
Compute the Kronecker product $X \otimes Y$ of

(1)
$$X = \begin{pmatrix} 1 & 2 \end{pmatrix}, Y = \begin{pmatrix} 3 & 4 \end{pmatrix}.$$

(2)
$$X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, Y = \begin{pmatrix} 3 & 4 \end{pmatrix}.$$

(3)
$$X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, Y = \begin{pmatrix} 5 & 6 \end{pmatrix}.$$

(4)
$$X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, Y = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}.$$

(5)
$$X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, Y = \begin{pmatrix} 5 & 6 \\ 7 & 8 \\ 9 & 10 \end{pmatrix}.$$

2. Understanding

Excercise 3 (The flip operator, ☕)**.** Find a matrix representation (in computational basis) of the linear operator $\mathbb{F} : \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d$ such that

$$\mathbb{F}(|a\rangle \otimes |b\rangle) = |b\rangle \otimes |a\rangle,$$

for every $|a\rangle, |b\rangle \in \mathbb{C}^d$. For convenience you may identify $\mathbb{C}^d \otimes \mathbb{C}^d \simeq \mathbb{C}^{d^2}$ using the lexicographic ordering of the computational basis on $\mathbb{C}^d \otimes \mathbb{C}^d$, i.e., such that

$$\begin{aligned} |\mathbf{1}_2\rangle &= |1\rangle \otimes |1\rangle \\ |\mathbf{2}_2\rangle &= |1\rangle \otimes |2\rangle \\ |\mathbf{3}_2\rangle &= |2\rangle \otimes |1\rangle \\ |\mathbf{4}_2\rangle &= |2\rangle \otimes |2\rangle, \end{aligned}$$

in the case $d = 2$.

Excercise 4 (Identifying operators,☕)**.** Show that for complex Euclidean spaces $\mathcal{H}_A, \mathcal{H}_B$ we have

$$B(\mathcal{H}_A) \otimes B(\mathcal{H}_B) \simeq B(\mathcal{H}_A \otimes \mathcal{H}_B).$$

Is it true that as vector spaces over $\mathbb{R}$ we have

$$B(\mathcal{H}_A)_{sa} \otimes B(\mathcal{H}_B)_{sa} \simeq B(\mathcal{H}_A \otimes \mathcal{H}_B)_{sa}, \tag{1}$$

for complex Euclidean spaces $\mathcal{H}_A, \mathcal{H}_B$? Is (1) true if $\mathcal{H}_A$ and $\mathcal{H}_B$ are real Euclidean spaces?

Excercise 5 (Characterizing positive operators, ☕☕)**.**
Let $P \in B(\mathcal{H})$ denote a linear operator on the complex Euclidean space $\mathcal{H}$.

(1) Show that the following are equivalent:
   (a) P is selfadjoint and has non-negative eigenvalues.
   (b) We have $\langle x|P|x\rangle \geqslant 0$ for every $|x\rangle \in \mathcal{H}$.
   (c) There exists a positive operator $Q \in B(\mathcal{H})$ such that $P = Q^2$.
   (d) There exists an operator $X \in B(\mathcal{H})$ such that $P = X^\dagger X$.
   (e) There exists an operator $Y \in B(\mathcal{H}, \mathcal{H}')$ for some Euclidean space $\mathcal{H}'$ such that $P = Y^\dagger Y$.

(2) Show that the operator Q in 3. is unique. This operator is called the *positive square root* of P. We will write $\sqrt{P}$ or $P^{1/2}$ to denote it.

Excercise 6 (The qubit,☕☕)**.** How can we visualize quantum states on the complex Euclidean space $\mathcal{H} = \mathbb{C}^2$?

(1) Consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that $\{\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3\}$ is an orthogonal basis of the Hilbert-Schmidt inner product space $B(\mathcal{H})$ and that their real span is $B(\mathcal{H})_{sa}$.

(2) Show that we have

$$\left(\sum_{i=1}^{3} x_i\sigma_i\right)^* \left(\sum_{i=1}^{3} x_i\sigma_i\right) = \|x\|_2^2 \mathbb{1}_2,$$

for any $x \in \mathbb{R}^3$. Conclude that the matrix

$$\rho_x = \frac{1}{2}\mathbb{1}_2 + \frac{1}{2}\sum_{i=1}^{3} x_i\sigma_i,$$

is a quantum state, i.e., positive and of unit trace, for any vector $x \in \mathbb{R}^3$ satisfying $\|x\|_2 \leqslant 1$.

(3) Show that every quantum state $\rho \in D(\mathbb{C}^2)$ can be written as $\rho = \rho_x$ for some $x \in \mathbb{R}^3$. Therefore, we can identify $D(\mathbb{C}^2)$ with the unit ball in the 2-norm. What are the pure states in this picture? What quantum state is at the center of the Bloch ball?

(4) Argue that $D(\mathbb{C}^3)$ is not isomorphic to the unit ball for any norm.

Excercise 7 (SICs, ☕☕☕)**.** A symmetrical informationally-complete PVM (also known as a SIC POVM) on $\mathbb{C}^d$ is a set of d^2 rank-1 projections

$$\{|\psi_1\rangle\langle\psi_1|, \ldots, |\psi_{d^2}\rangle\langle\psi_{d^2}|\} \subset \text{Proj}\left(\mathbb{C}^d\right)$$

such that

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{d\delta_{ij} + 1}{d+1},$$

for any $i, j \in \{1, \ldots, d^2\}$. Construct a SIC POVM on $\mathbb{C}^2$.