# EXERCISES 2

## 1. Training

**Excercise 1** (Quantum states)**.**
Which of the following matrices are quantum states in $D(\mathbb{C}^2 \otimes \mathbb{C}^2)$?

(1)

$$\rho_{AB} = \frac{1}{6}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(2)

$$\sigma_{AB} = \frac{1}{6}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(3)

$$\tau_{AB} = \frac{1}{12}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix}.$$

For any of the matrices above, compute both partial traces $\text{Tr}_A$ and $\text{Tr}_B$.

**Excercise 2** (Entanglement in pure states)**.**
Consider the Euclidean space $\mathcal{H} = \mathbb{C}^2$ and the following vectors in $\mathcal{H} \otimes \mathcal{H}$:

(1) $|\psi_1\rangle = \frac{1}{2}\,\text{vec}\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)$.

(2) $|\psi_2\rangle = \frac{1}{2}\,\text{vec}\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)$.

(3) $|\psi_3\rangle = \frac{1}{2}\,\text{vec}\left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right)$.

(4) $|\psi_4\rangle = \frac{1}{2\sqrt{2}}\left((|1\rangle + |2\rangle) \otimes (|1\rangle + |2\rangle) - (|1\rangle - |2\rangle) \otimes (|1\rangle - |2\rangle)\right)$.

Which of the corresponding pure states $|\psi_i\rangle\langle\psi_i|$ are entangled?

**Excercise 3** (Monogamy of entanglement)**.** Show that all purifications of a pure state $\rho_A = |\psi\rangle\langle\psi| \in D(\mathcal{H}_A)$ are of the form

$$\rho_{AE} = |\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi|.$$

Now consider a composite quantum system 'AB' with tensor product state space $\mathcal{H}_A \otimes \mathcal{H}_B$ and assume that it is in a pure entangled quantum state $|\psi_{AB}\rangle\langle\psi_{AB}| \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$. Argue that there is no third quantum system that is entangled with the composite system 'AB'.

**Excercise 4.** Is the matrix transpose $\vartheta_d : B(\mathbb{C}^d) \to B(\mathbb{C}^d)$ given by $\vartheta_d(X) = X^T$ (in the computational basis) a quantum channel?

## 2. Understanding

**Excercise 5** (Examples of quantum channels,)**.** Consider an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$. Show that the map $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ given by

$$T(X) = \mathrm{Tr}_E \left[ V X V^\dagger \right],$$

is a quantum channel.

**Excercise 6** (Purifications and reduced density matrices,)**.** Consider a pure quantum state $|\psi_{AB}\rangle\langle\psi_{AB}| \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ and denote by $\rho_A$ and $\rho_B$ its two marginals. Show the following statements:

(1) The spectra of $\rho_A$ and $\rho_B$ are the same. What is the relationship between the eigenvalues of these operators and the Schmidt coefficients of $|\psi_{AB}\rangle$?

(2) There exists an operator $X \in B(\mathcal{H}_A, \mathcal{H}_B)$ such that

$$\rho_A = X^T \overline{X} \text{ and } \rho_B = X X^\dagger.$$

(3) There is a complex Euclidean space $\mathcal{H}_E$ with $\dim(\mathcal{H}_E) = \mathrm{rk}\,(\rho_A)$ and a purification $|\phi_{AE}\rangle$ of $\rho_A$ such that

$$|\psi_{AB}\rangle = (\mathbb{1}_{\mathcal{H}_A} \otimes V)|\phi_{AE}\rangle,$$

for some isometry $V : \mathcal{H}_E \to \mathcal{H}_B$. Note that such an isometry exists for any purification of $\rho_A$.

**Excercise 7** (The realignment criterion,)**.** For complex Euclidean spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ we define the realignment map $R : B(\mathcal{H}_A \otimes \mathcal{H}_B) \to B\left(\mathcal{H}_B \otimes \mathcal{H}_B, \mathcal{H}_A \otimes \mathcal{H}_A\right)$ by

$$R\left(|i_A\rangle\langle j_A| \otimes |k_B\rangle\langle l_B|\right) = |i_A\rangle\langle k_B| \otimes |j_A\rangle\langle l_B|,$$

and extended linearly.

(1) Show that

$$R(X \otimes Y) = |\,\mathrm{vec}(X^T)\rangle\langle\mathrm{vec}(Y^T)|,$$

for any $X \in B(\mathcal{H}_A)$ and $Y \in B(\mathcal{H}_B)$.

(2) Assume that $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a separable quantum state. Show that

$$\|R(\rho_{AB})\|_1 \leqslant 1.$$

Here, $\|\cdot\|_1$ denotes the 1-norm, i.e., the norm given by

$$\|A\|_1 = \mathrm{Tr}\left[\sqrt{A^\dagger A}\right],$$

for any $A \in B(\mathcal{H})$, which also equals the sum of singular values of $A$.

(3) Show that the maximally entangled state $\omega_{\mathcal{H}}$ satisfies

$$\|R(\omega_{\mathcal{H}})\|_1 > 1,$$

showing that $\omega_{\mathcal{H}}$ is entangled.

**Excercise 8** (Freedom in extensions,)**.** Let $\mathcal{H}_A, \mathcal{H}_E$ and $\mathcal{H}_{E'}$ denote complex Euclidean spaces. Consider a purification $|\psi_{AE}\rangle\langle\psi_{AE}| \in D(\mathcal{H}_A \otimes \mathcal{H}_E)$ of a quantum state $\rho_A \in D(\mathcal{H}_A)$ and a quantum state $\sigma_{AE'} \in D(\mathcal{H}_A \otimes \mathcal{H}_{E'})$ such that $\mathrm{Tr}_{E'}\left[\sigma_{AE'}\right] = \rho_A$. Show that there exists a quantum channel $S : B(\mathcal{H}_E) \to B(\mathcal{H}_{E'})$ such that

$$\left(\mathrm{id}_A \otimes S\right)\left(|\psi_{AE}\rangle\langle\psi_{AE}|\right) = \sigma_{AE'}.$$

*Hint:* Start by reducing the problem to the case, where $\sigma_{AE'}$ is a pure state as well.