EXERCISES 3

1. Training

Excercise 1. Show that a linear map $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is completely positive if and only if $(\mathrm{id}_A \otimes T) : B(\mathcal{H}_A \otimes \mathcal{H}_A) \to B(\mathcal{H}_A \otimes \mathcal{H}_B)$ is positive.

Excercise 2 (The amplitude damping channel)**.** For $\gamma \in [0,1]$ we define the *amplitude-damping channel* $N_\gamma : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ by

$$N_\gamma \left[\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right] = \begin{pmatrix} x_{11} + \gamma x_{22} & \sqrt{1-\gamma} x_{12} \\ \sqrt{1-\gamma} x_{21} & (1-\gamma) x_{22} \end{pmatrix}.$$

(1) Compute the Choi matrix of N_γ. Is N_γ a quantum channel?
(2) Compute a Choi-Kraus representation of N_γ.
(3) Compute a Stinespring dilation of N_γ.

Using the criterion from Exercise 7, show that the amplitude-damping channel is extremal in the set of all quantum channels for every parameter $\gamma \in [0,1]$. Is this surprising?

Excercise 3. For a linear map $S : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ we define its adjoint $S^* : B(\mathcal{H}_B) \to B(\mathcal{H}_A)$ as the map satisfying

$$\langle Y, S(X) \rangle_{HS} = \mathrm{Tr}\left[Y^\dagger S(X) \right] = \mathrm{Tr}\left[(S^*(Y))^\dagger X \right] = \langle S^*(Y), X \rangle_{HS}.$$

Show that the dual of a quantum channel T is a unital completely positive map T^*, i.e., a completely positive map T^* satisfying $T^*(\mathbb{1}_{\mathcal{H}_B}) = \mathbb{1}_{\mathcal{H}_A}$. Can you express such a map T^* by taking the dual of the Stinespring dilation?

2. Understanding

Excercise 4 (The Weyl-Heisenberg system, ☕)**.** The goal of this exercise is to construct the so-called *Weyl-Heisenberg system*, an orthogonal basis consisting of unitary operators for the Hilbert-Schmidt inner product space $B(\mathbb{C}^d)$. This system generalizes the Pauli matrices to higher dimensions.

(1) Consider the cyclic shift operator $S : \mathbb{C}^d \to \mathbb{C}^d$ given by

$$S|i\rangle = |i+1\rangle,$$

on the computational basis $\{|0\rangle, \dots, |d-1\rangle\}$, and where addition is mod d. Show that the powers S^k for $k \in \{0, \dots, d-1\}$ are orthogonal with respect to the Hilbert-Schmidt inner product $\langle X, Y \rangle = \mathrm{Tr}\left[X^\dagger Y\right]$.

(2) For the dth root of unity $\omega_d = \exp(\frac{2\pi i}{d})$ consider the diagonal matrix

$$D = \begin{pmatrix} 1 & & & & \\ & \omega_d & & & \\ & & \omega_d^2 & & \\ & & & \ddots & \\ & & & & \omega_d^{d-1} \end{pmatrix} \in B(\mathbb{C}^d).$$

Show that the powers D^l for $l \in \{0, \dots, d-1\}$ are orthogonal with respect to the Hilbert-Schmidt inner product.

(3) Finally, define unitary operators $U_{kl} = S^k D^l$ and show that they form an orthogonal basis for the Hilbert-Schmidt inner product space $B(\mathbb{C}^d)$. What operators do you get for $d = 2$?

Excercise 5 (The Moore-Penrose pseudo-inverse, ☕)**.** Let $\mathcal{H}$ denote a complex Euclidean space and consider an operator $X \in B(\mathcal{H})$. If $X = USV^\dagger$ is the singular value decomposition of X, then we define the *Moore-Penrose pseudo-inverse* of X as

$$X^{-1} = VS^{-1}U^\dagger,$$

where we set

$$S^{-1} = \begin{pmatrix} \frac{1}{s_1} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{s_k} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

for the non-zero singular values $s_1, \ldots, s_k$. Show the following:

(1) $XX^{-1}X = X$.
(2) $X^{-1}XX^{-1} = X^{-1}$.
(3) XX^{-1} and $X^{-1}X$ are selfadjoint.

Furthermore, show that in the case where X is selfadjoint, then $X^{-1}X = XX^{-1}$ and this operator coincides with the orthogonal projection onto the range of X.

Excercise 6 (Ensemble representation theorem,☕☕)**.** For a complex Euclidean space of dimension $\dim(\mathcal{H}) = d$ consider two sets $\{|\psi_n\rangle\}_{n=1}^N \subset \mathcal{H}$ and $\{|\phi_k\rangle\}_{k=1}^K \subset \mathcal{H}$ of vectors. We aim to show that

$$\sum_{n=1}^N |\psi_n\rangle\langle\psi_n| = \sum_{k=1}^K |\phi_k\rangle\langle\phi_k|.$$

if and only if there exists a unitary operator $U \in \mathcal{U}\left(\mathbb{C}^N\right)$ such that

$$|\psi_n\rangle = \sum_{k=1}^K U_{nk}|\phi_k\rangle,$$

where the smaller set of vectors is extended by zero vectors. To show this result apply the following steps, or find your own proof:

(1) Show the easy direction.
(2) Consider a set of orthonormal vectors $\{|e_k\rangle\}_{k=1}^K \subset \mathcal{H}$ and let $\{|\psi_n\rangle\}_{n=1}^N \subset \mathcal{H}$ for $N \geqslant K$ satisfy

$$\sum_{n=1}^N |\psi_n\rangle\langle\psi_n| = \sum_{k=1}^K |e_k\rangle\langle e_k|.$$

Show that

$$\{\left(|e_k\rangle\}_{k=1}^K\right)^\perp \subseteq \left(\{|\psi_n\rangle\}_{n=1}^N\right)^\perp,$$

where $\mathcal{S}^\perp$ denotes the orthogonal complement of a set $\mathcal{S} \subseteq \mathcal{H}$. Then, construct a unitary $U \in \mathcal{U}\left(\mathbb{C}^N\right)$ such that

$$\sum_{n=1}^N U_{kn}|\psi_n\rangle = |e_k\rangle,$$

and

$$\sum_{k=1}^{K} \overline{U_{kn}} |e_k\rangle = |\psi_n\rangle,$$

for each $n \in \{1, \dots, N\}$ and $k \in \{1, \dots, K\}$.

(3) Consider next the case, where $\{|e_k\rangle\}_{k=1}^{K} \subset \mathcal{H}$ are the eigenvectors corresponding to non-zero eigenvalues $\{\lambda_1, \dots, \lambda_K\}$ of the operator

$$A = \sum_{n=1}^{N} |\psi_n\rangle\langle\psi_n|.$$

Construct a unitary operator $U \in \mathcal{U}\left(\mathbb{C}^N\right)$ such that

$$\sum_{n=1}^{N} U_{kn} |\psi_n\rangle = \sqrt{\lambda_k} |e_k\rangle,$$

and

$$\sum_{k=1}^{K} \overline{U_{kn}} \sqrt{\lambda_k} |e_k\rangle = |\psi_n\rangle,$$

for each $n \in \{1, \dots, N\}$ and $k \in \{1, \dots, K\}$.

(4) Finally, combine the insights of the previous points to prove the theorem stated in the beginning of this exercise.

(5) Relate this result to the Kraus representation of a completely positive map and show the theorem from the lecture notes.

Excercise 7 (Extremal quantum channels, ☕☕☕)**.** Let $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ denote a quantum channel with Choi-Kraus representation $T = \sum_{i=1}^{N} \mathrm{Ad}_{K_i}$ such that the set $\{K_i\}_{i=1}^{N} \subset B(\mathcal{H}_A, \mathcal{H}_B)$ is linearily independent. Show that the quantum channel T is extremal in the set of all quantum channels if and only if the set

$$\{K_i^\dagger K_j\}_{i,j=1}^{N} \tag{1}$$

is linearily independent in $B(\mathcal{H}_A)$. For this follow, the steps below:

(1) Assume first that the set in (1) is linearily independent, and assume that

$$T = (1-p)S_1 + pS_2$$

for some $p \in (0,1)$ and quantum channels $S_1, S_2 : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$. Express the Kraus operators of S_1 in terms of $\{K_i\}_{i=1}^{N}$ and show that $S_1 = T$.

(2) Now assume that T is extremal and assume that

$$\sum_{i,j=1}^{N} C_{ij} K_i^\dagger K_j = 0,$$

for some $C \in B(\mathbb{C}^N)$. Show that this condition can be expressed as

$$V^\dagger \left(\mathbb{1}_{\mathcal{H}_B} \otimes C\right) V = 0,$$

where V is the isometry obtained by stacking the Kraus operators $\{K_i\}_i$ on top of each other.

(3) Show that without loss of generality the operator C can be assumed to be selfadjoint and such that its operator norm satsfies $\|C\| \leqslant 1$.

(4) Finally, construct two completely positive maps

$$S_\pm^*(X) = V^\dagger \left(X \otimes (\mathbb{1}_{\mathbb{C}^N} \pm C)\right) V,$$

and show that they are unital. Furthermore, show that

$$T = \frac{1}{2}\left(S_+ + S_-\right),$$

implying that $T = S_+ = S_-$. Conclude from this that $C = 0$.