EXERCISES 4

Excercise 1. Check that $\mu : \{1,2\} \to B(\mathbb{C}^2)^+$ given by

$$\mu(1) = \frac{1}{4}\begin{pmatrix} 2+\sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2-\sqrt{2} \end{pmatrix}, \quad \mu(2) = \frac{1}{4}\begin{pmatrix} 2-\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 2+\sqrt{2} \end{pmatrix},$$

defines a POVM.

Excercise 2. A POVM $\mu : \Sigma \to B(\mathcal{H})^+$ is called *informationally complete* if every quantum state $\rho \in D(\mathcal{H})$ can be uniquely determined from the probabilities $p(x) = \mathrm{Tr}\left[\mu(x)\rho\right]$. Show that for an informationally complete POVM the outcome alphabet has size $|\Sigma| \geqslant d^2$.

1. Training

2. Understanding

Excercise 3 (Superdense coding, ☕)**.** Consider two researchers named Alice and Bob who each have a laboratory. Alice's laboratory contains a quantum system labelled 'A' and Bob's laboratory contains a quantum system labelled 'B'. Each of these quantum systems is a qubit, i.e., their state space H_A and H_B are equal to $\mathbb{C}^2$. Initially, 'A' and 'B' are in the maximally entangled pure state, i.e., their state is given by $\omega_{AB} = \frac{1}{2}|\Omega_{AB}\rangle\langle\Omega_{AB}| \in D\left(H_A \otimes H_B\right)$ with

$$|\Omega_{AB}\rangle = \frac{1}{\sqrt{2}}\sum_{i=1}^{2} |i_A\rangle \otimes |i_B\rangle.$$

Superdense coding is a protocol by which Alice can send 2 bits of classical information by transferring a qubit to Bob. Proceed as follows:

(1) Consider the Pauli matrices written as

$$\sigma_{00} = \mathbb{1}_2, \quad \sigma_{01} = \sigma_x, \quad \sigma_{10} = \sigma_y, \quad \sigma_{11} = \sigma_z.$$

Show that the vectors

$$|\psi_{ij}\rangle = \left(\mathbb{1}_2 \otimes \sigma_{ij}\right)|\Omega_{AB}\rangle$$

define an orthonormal basis of $H_A \otimes H_B$. This orthonormal basis is called the "Bell basis".

(2) Find quantum channels $T_{ij} : B(H_A) \to B(H_A)$ such that

$$\mathrm{Tr}\left[|\psi_{kl}\rangle\langle\psi_{kl}|\left(T_{ij} \otimes \mathrm{id}_2\right)(\omega_{AB})\right] = \delta_{ki}\delta_{lj}.$$

for any $k, l, i, j \in \{0,1\}$.

(3) Alice can use the quantum channels T_{ij} to "encode" 2 bits (i,j). If Alice transfers her qubit to Bob, then he can retrieve the message (i,j) by a measurement. How does this work in detail?

Excercise 4 (Quantum teleportation, ☕☕)**.** Consider again the two researchers Alice and Bob who each have a laboratory. Alice's laboratory contains two quantum systems labelled 'A' and '$\tilde{A}$' and Bob's laboratory contains a quantum system labelled 'B'. Each of these quantum systems is a qubit. Initially, 'A' and 'B' are in a maximally entangled pure state (see previous exercise). Assume furthermore, that the system '$\tilde{A}$' is initially in some quantum state $\rho_{\tilde{A}} \in D(\mathcal{H}_{\tilde{A}})$ unknown to Alice and Bob. Quantum teleportation is a protocol that lets Alice send this unknown state to Bob by performing a measurement and sending the measurement outcome to Bob. Proceed as follows:

(1) Recall the Bell basis $\{|\psi_{ij}\rangle\}_{ij} \subset \mathbb{C}^2 \otimes \mathbb{C}^2$ from the previous exercise. Compute the probability of obtaining the outcome $(k,l) \in \{0,1\}^2$ when measuring the subsystems '$\tilde{A}A$' of the quantum state
$$\sigma_{ini} = \rho_{\tilde{A}} \otimes \omega_{AB},$$
using the von-Neumann measurement $\{|\psi_{ij}\rangle\langle\psi_{ij}|\}_{i,j}$ defined via the Bell basis.

(2) What is the post-measurement state σ^{ij}_{post} after obtaining outcome (i,j) in the measurement of (1)? **Hint:** You can either consider a destructive measurement, or some instrument corresponding to the Bell measurement. Does the choice of instrument matter? It is very helpful to draw a diagram to see what is going on here!

(3) If Alice communicates the measurement outcome (i,j) of the Bell measurement to Bob, what quantum channel can he apply so that his system is in the final state $\rho_{\tilde{A}}$.

(4) How can we extend this protocol to quantum systems with dimension $d > 2$?

Excercise 5 (Quantum steering, ☕☕☕)**.** Let $\rho_A \in D(\mathcal{H}_A)$ denote a quantum state with purification $|\psi_{AB}\rangle\langle\psi_{AB}| \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$, and consider a decomposition $\rho_A = \sum_{n=1}^N p_n \rho_n$ with quantum states $\rho_n \in D(\mathcal{H}_A)$ and a probability distribution $p \in \mathcal{P}(\{1,\dots,N\})$. We aim to construct an instrument $\{T_n\}_{n=1}^N$ with $T_n : B(\mathcal{H}_B) \to B(\mathcal{H}_B)$ such that
$$\mathrm{Tr}_B\left[(\mathrm{id}_A \otimes T_n)(|\psi_{AB}\rangle\langle\psi_{AB}|)\right] = p_n\rho_n,$$
for any $n \in \{1,\dots,N\}$. Follow the steps below:

(1) Argue that without loss of generality we can assume $|\psi_{AB}\rangle = \mathrm{vec}\left(\sqrt{\rho_A}\right)$.

(2) Show that
$$\mathrm{Tr}_B\left[(\mathrm{id}_A \otimes T)\left(\mathrm{vec}\left(\sqrt{\rho}\right)\mathrm{vec}\left(\sqrt{\rho}\right)^\dagger\right)\right] = \sqrt{\rho}\left[T^*\left(\mathbb{1}_{H_B}\right)\right]^T\sqrt{\rho},$$
for any completely positive map $T : B(H_B) \to B(H_B)$ and any $\rho \in D(\mathcal{H}_A)$.

(3) Find operators $K_n \in B(\mathcal{H}_B)$ for every $n \in \{1,\dots,N\}$ such that
$$\sqrt{\rho_A}(K_n^\dagger K_n)^T\sqrt{\rho_A} = p_n\rho_n,$$
and such that $\sum_{n=1}^N K_n^\dagger K_n = \mathbb{1}_{\mathcal{H}_B}$.

Hint: Consider first the case where ρ_A is invertible and use the Moore-Penrose pseudoinverse for the general case.

(4) Construct an instrument $\{T_n\}_{n=1}^N$ with $T_n : B(\mathcal{H}_B) \to B(\mathcal{H}_B)$ such that
$$\mathrm{Tr}_B\left[(\mathrm{id}_A \otimes T_n)(|\psi_{AB}\rangle\langle\psi_{AB}|)\right] = p_n\rho_n,$$
for any $n \in \{1,\dots,N\}$.

(5) An interpretation of this result is that it is impossible to construct a so-called *mixed state analyzer*, a hypothetical device that can distinguish different decompositions $\rho_A = \sum_{n=1}^N p_n\rho_n$ and $\rho_A = \sum_{m=1}^M q_m\sigma_m$ of the same quantum state. Can you explain why?