EXERCISES 5

1. Training

Excercise 1. Consider a convex function $f : \mathcal{V} \to \mathbb{R}$ on a real vector space $\mathcal{V}$, i.e., a function satisfying

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y),$$

for every $x, y \in \mathcal{V}$ and every $\lambda \in [0,1]$. For a convex body B show that

$$\sup_{x \in B} f(x) = \max_{x \in \mathrm{Ext}(B)} f(x),$$

i.e., the supremum of f over B is attained in an extreme point of B.

Excercise 2. Consider the cone $C = \{(x,y,0) \in \mathbb{R}^3 \ : \ x, y \geqslant 0\}$ in $\mathbb{R}^3$. Compute the dual cone C^*. Is C (or C^*) pointed? Is C^* (or C) generating?

2. Understanding

Excercise 3 (Properties of cones, ☕)**.** Consider two closed cones $C_1, C_2 \subset \mathcal{V}$ in a real vector space $\mathcal{V}$.

(1) Show that if $C_1 \subseteq C_2$, then $C_1^* \supseteq C_2^*$.
(2) Show that $(C_1 \cap C_2)^* = \overline{(C_1^* \vee C_2^*)}$.

Excercise 4 (Classicality of the simplex cone, ☕)**.** Consider a proper cone $C \subset \mathcal{V}$ in a real vector space $\mathcal{V}$ and the cone $\mathbb{R}^N_+$ entrywise positive vectors in R^N. Show that

$$\mathbb{R}^N_+ \otimes_{\min} C = \mathbb{R}^N_+ \otimes_{\max} C.$$

Excercise 5 (Properties of tensor products, ☕☕)**.** Consider proper cones $C_A \subset \mathcal{V}_A$ and $C_B \subset \mathcal{V}_B$ belonging to real Euclidean spaces $\mathcal{V}_A$ and $\mathcal{V}_B$, respectively.

(1) Show that both $C_A \otimes_{\min} C_B$ and $C_A \otimes_{\max} C_B$ are proper cones in $\mathcal{V}_A \otimes \mathcal{V}_B$.
(2) Show that $(C_A \otimes_{\min} C_B)^* = C_A^* \otimes_{\max} C_B^*$.
(3) Show that

$$C_A \otimes_{\min} C_B \subseteq C_A \otimes_{\max} C_B.$$

Excercise 6 (Entanglement manipulation, ☕☕☕)**.** Let $\mathcal{H}_{A_1}, \mathcal{H}_{A_2}, \mathcal{H}_{B_1}, \mathcal{H}_{B_2}$ denote complex Euclidean spaces. A quantum channel

$$T : B(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}) \to B(\mathcal{H}_{A_2} \otimes H_{\mathcal{B}_2})$$

is called a *separable*[1] *quantum channel* if it can be written as

$$T = \sum_{n=1}^{N} S_A^{(n)} \otimes R_B^{(n)},$$

for completely positive maps $S_A^{(n)} : B(\mathcal{H}_{A_1}) \to B(\mathcal{H}_{A_2})$ and $R_B^{(n)} : B(\mathcal{H}_{B_1}) \to B(\mathcal{H}_{B_2})$. Let us denote the set of separable channels by

$$\mathrm{SepC}\left((\mathcal{H}_{A_1}, \mathcal{H}_{B_1}) \to (\mathcal{H}_{A_2}, \mathcal{H}_{B_2})\right).$$

(1) Show that $T(\rho_{AB}) \in \mathrm{Sep}(\mathcal{H}_{A_2}, \mathcal{H}_{B_2})$ for any $\rho_{AB} \in \mathrm{Sep}(\mathcal{H}_{A_1}, \mathcal{H}_{B_1})$ and any $\mathrm{SepC}\left((\mathcal{H}_{A_1}, \mathcal{H}_{B_1}) \to (\mathcal{H}_{A_2}, \mathcal{H}_{B_2})\right)$. The separable states are preserved under separable operations!

[1] Do not confuse this with separable quantum states!

(2) Using the teleportation protocol, construct a separable quantum channel

$$T \in \mathrm{SepC}\left((\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2) \to (\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2)\right),$$

such that

$$T(\rho \otimes \omega_{AB}) = \frac{1}{4}(\mathbb{1}_{\mathbb{C}^2} \otimes \mathbb{1}_{\mathbb{C}^2}) \otimes \rho \in D(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2),$$

for any quantum state $\rho \in D(\mathbb{C}^2)$ and where $\omega_{AB} \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$ denotes the (normalized) maximally entangled state. How does this result generalize to dimensions $d > 2$.

(3) Show that any quantum states $\rho \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ can be written as

$$\rho = T(\omega_d),$$

for some separable quantum channel $T \in \mathrm{SepC}\left((\mathbb{C}^d, \mathbb{C}^d) \to (\mathbb{C}^d, \mathbb{C}^d)\right)$, and where $\omega_d \in D(\mathbb{C}^d \otimes \mathbb{C}^d)$ denotes the normalized maximally entangled state. This is why ω_d is called *maximally* entangled.

(4) Assume that an experimental physicist comes to you and claims that a certain quantum channel $T : B(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1}) \to B(\mathcal{H}_{A_2} \otimes H_{\mathcal{B}_2})$ can be implemented by using two high-end quantum laboratories 'A' and 'B' that are far apart and assisted by classical messages sended back and forth between them. Convince her that the quantum channel T is a separable operation.