EXERCISES 5

1. Training

Excercise 1. Using the positive partial transpose criterion, show that the quantum state

$$\rho_{AB} = \frac{1}{3}\begin{pmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & 1/2 \end{pmatrix} \in D(\mathbb{C}^2 \otimes \mathbb{C}^2)$$

is separable.

2. Understanding

Excercise 2 (A modified singular value decomposition, ☕)**.** Let O(3) denote the orthogonal group on $\mathbb{R}^3$, i.e., the matrices $U \in B(\mathbb{R}^3)$ such that $U^T U = \mathbb{1}_{\mathbb{R}^3}$ and SO(3) denote the group of 3-dimensional rotation matrices, i.e., orthogonal matrices with determinant 1. Consider a matrix $M \in B(\mathbb{R}^3)$ (i.e., with real entries!). Show that there exist $R_1, R_2 \in \mathrm{SO}(3)$ such that

$$M = R_1 D R_2,$$

with $D \in B(\mathbb{R}^3)$ diagonal and with either positive or negative entries.

(1) Show that there is a singular value decomposition $M = USV$ with orthogonal matrices $U, V \in \mathrm{O}\,(3)$.
(2) Modify the orthogonal matrices in (1) prove the modified singular value decomposition from above.

Excercise 3 (Diagonalizing positive unital and trace-preserving qubit maps,☕)**.** We want to diagonalize positive linear maps $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ satisfying $P(\mathbb{1}_{\mathbb{C}^2}) = \mathbb{1}_{\mathbb{C}^2}$ and $\mathrm{Tr}\,[P(X)] = \mathrm{Tr}\,[X]$ for any $X \in B(\mathbb{C}^2)$. For this we recall (from a previous exercise) that there is a 1-to-1 correspondence between the quantum states $D(\mathbb{C}^2)$ and the Euclidean unit ball $B_1(0) \subset \mathbb{R}^3$ (the Bloch ball) given by

$$\rho = \frac{1}{2}\left(\mathbb{1}_{\mathbb{C}^2} + w \cdot \sigma\right) \in D(\mathbb{C}^2) \quad \longleftrightarrow \quad w \in B_1(0),$$

where we use the shorthand

$$w \cdot \sigma = \sum_{i=1}^{3} w_i \sigma_i,$$

with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will sometimes write $\sigma_0 = \mathbb{1}_{\mathbb{C}^2}$ and we recall that the real span of $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ is $B(\mathbb{C}^2)_{sa}$ and the complex span is $B(\mathbb{C}^2)$. Proceed as follows:

(1) Consider a positive map $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ satisfying $P(\mathbb{1}_{\mathbb{C}^2}) = \mathbb{1}_{\mathbb{C}^2}$ and $\mathrm{Tr}\,[P(X)] = \mathrm{Tr}\,[X]$ for any $X \in B(\mathbb{C}^2)$. Show that $M_{ij} = \frac{1}{2}\,\mathrm{Tr}\,[\sigma_j P(\sigma_i)] \in \mathbb{R}$ for every $i, j \in \{1, 2, 3\}$ and that

$$P\left(\frac{1}{2}\left(\mathbb{1}_{\mathbb{C}^2} + w \cdot \sigma\right)\right) = \frac{1}{2}\left(\mathbb{1}_{\mathbb{C}^2} + (Mw) \cdot \sigma\right),$$

for any $w \in \mathbb{R}^3$, where we introduced $M \in B(\mathbb{R}^3)$ with entries M_{ij}.

(2) Consider the unitary
$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$
Find an $R \in \mathrm{SO}(3)$ such that
$$\mathrm{Ad}_U\left(\frac{1}{2}\left(\mathbb{1}_{\mathbb{C}^2} + w \cdot \sigma\right)\right) = \frac{1}{2}\left(\mathbb{1}_{\mathbb{C}^2} + (Rw) \cdot \sigma\right).$$

(3) Now, consider a general $R \in \mathrm{SO}(3)$, i.e., corresponding to a 3-dimensional rotation of the ball $B_1(0)$. Show that there exists a $U \in \mathcal{U}\left(\mathbb{C}^2\right)$ such that
$$\mathrm{Ad}_U\left(\frac{1}{2}\left(\mathbb{1}_{\mathbb{C}^2} + w \cdot \sigma\right)\right) = \frac{1}{2}\left(\mathbb{1}_{\mathbb{C}^2} + (Rw) \cdot \sigma\right).$$
Hint: Start with rotations around the coordinate axes. Then, use that a general $R \in \mathrm{SO}(3)$ can be written as the product three rotations around the coordinate axes with certain rotation angles (sometimes called *Euler angles*).

(4) Consider a positive map $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ satisfying $P(\mathbb{1}_{\mathbb{C}^2}) = \mathbb{1}_{\mathbb{C}^2}$ and $\mathrm{Tr}\left[P(X)\right] = \mathrm{Tr}\left[X\right]$ for any $X \in B(\mathbb{C}^2)$. Use Exercise 2 to show that there are unitary matrices $U_1, U_2 \in \mathcal{U}\left(\mathbb{C}^2\right)$ satisfying
$$\left(\mathrm{Ad}_{U_1} \circ P \circ \mathrm{Ad}_{U_2}\right)(\sigma_i) = \lambda_i \sigma_i,$$
for any $i \in \{0, 1, 2, 3\}$ and some real numbers $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\lambda_0 = 1$.

Excercise 4 (Visualizing positive maps,☕)**.** For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, consider the trace-preserving map $P_\lambda : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ such that $P_\lambda(\mathbb{1}_{\mathbb{C}^2}) = \mathbb{1}_{\mathbb{C}^2}$ and
$$P_\lambda\left(\sigma_i\right) = \lambda_i \sigma_i$$
for $i \in \{1, 2, 3\}$.

(1) For which $\lambda \in \mathbb{R}^3$ is P_λ positive? Visualize this set.

(2) For which $\lambda \in \mathbb{R}^3$ is P_λ completely positive? Visualize this set.
Hint: Compute the Choi-Jamiolkowski operator of P_λ. Use that $\sigma_i \otimes \sigma_i$ and $\sigma_j \otimes \sigma_j$ commute and are simultaneously diagonalized by the Bell basis.

(3) For which $\lambda \in \mathbb{R}^3$ is $P_\lambda = \vartheta \circ T$ for the transpose $\vartheta : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ and a completely positive map $T : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$? Visualize this set.

(4) Using your visualizations and Exercise 3, show that any positive map $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ satisfying $P(\mathbb{1}_{\mathbb{C}^2}) = \mathbb{1}_{\mathbb{C}^2}$ and $\mathrm{Tr}\left[P(X)\right] = \mathrm{Tr}\left[X\right]$ for any $X \in B(\mathbb{C}^2)$ can be written as
$$P = T_1 + \vartheta \circ T_2,$$
for completely positive maps $T_1, T_2 : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ and the transpose map $\vartheta : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$.