EXERCISES 16

Excercise 1 (Channels with vanishing classical capacity)**.** Let $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ denote a quantum channel. Show that $C(T) = 0$ if and only if T is constant, i.e., there exists a quantum state $\sigma \in D(\mathcal{H}_B)$ such that $T(X) = \mathrm{Tr}[X]\,\sigma$.

Excercise 2. Consider the depolarizing channel $D_\lambda : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ with parameter $\lambda \in [0,1]$ given by

$$D_\lambda(X) = \lambda\,\mathrm{Tr}[X]\,\frac{\mathbb{1}_2}{2} + (1-\lambda)X.$$

We want to evaluate $\chi(D_\lambda)$, which can be shown to be equal to the classical capacity of the depolarizing channel $C(D_\lambda)$.

(1) Show that $D_\lambda \circ \mathrm{Ad}_{\sigma_i} = \mathrm{Ad}_{\sigma_i} \circ D_\lambda$ for any Pauli matrix σ_i.

(2) Consider an ensemble $\{p(x), \rho_x\}$ with $p \in \mathcal{P}(\Sigma)$ and $\rho_x \in D(\mathbb{C}^2)$. Show that

$$\chi\left(\{\{p(x), D_\lambda(\rho_x)\}\}\right) \leqslant \chi\left(\{\{q(x,i), D_\lambda(\sigma_{x,i})\}\}\right),$$

where $q(x,i) = p(x)/4$ and $\sigma_{x,i} = \mathrm{Ad}_{\sigma_i}(\rho_x)$.

(3) Use

$$\frac{1}{4}\sum_{i=1}^{4} \mathrm{Ad}_{\sigma_i}(X) = \mathrm{Tr}[X]\,\frac{\mathbb{1}_2}{2},$$

to show that

$$\chi(D_\lambda) = 1 - \min_{\rho\in D(\mathbb{C}^2)} H\left(D_\lambda(\rho)\right).$$

(4) Verify that $\min_{\rho\in D(\mathbb{C}^2)} H\left(D_\lambda(\rho)\right)$ is attained in any pure quantum state leading to the formula

$$\chi(D_\lambda) = 1 + \frac{\lambda}{2}\log\left(\frac{\lambda}{2}\right) + \left(\frac{\lambda}{2} + (1-\lambda)\right)\log\left(\frac{\lambda}{2} + (1-\lambda)\right).$$

(5) Think about how this result generalizes to $d \geqslant 3$.

Excercise 3 (Entanglement breaking channels)**.** A quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is called *entanglement breaking* if its normalized Choi operator is a separable quantum state.

(1) Show that a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is entanglement breaking if and only if it can be written as

$$T(X) = \sum_{i=1}^{N} \mathrm{Tr}[A_i X]\,B_i,$$

for some $N \in \mathbb{N}$ and positive semidefinite operators $A_i \in B(\mathcal{H}_A)^+$ and $B_i \in B(\mathcal{H}_B)^+$.

(2) Show that a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is entanglement breaking if and only if it admits a Kraus decomposition with rank-1 Kraus operators.

(3) Show that a quantum channel $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is entanglement breaking if and only if $(\mathrm{id}_E \otimes T)(\rho_{EA})$ is separable for any quantum state $\rho_{EA} \in D(\mathcal{H}_E \otimes \mathcal{H}_A)$ for any complex Euclidean space $\mathcal{H}_E$.

Excercise 4 (Capacity of entanglement breaking channels)**.** Let $T : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ denote an entanglement breaking channel (see previous exercise). We will show that

$$\chi(T \otimes S) = \chi(T) + \chi(S),$$

for any quantum channel $S : B(\mathcal{H}_C) \to B(\mathcal{H}_D)$. By the HSW-theorem this implies that the classical capacity of T equals the Holevo-information, i.e., $C(T) = \chi(T)$.

(1) Consider ensembles $\{p(x), \rho_x\}_{x\in\Sigma_1}$ and $\{q(y), \sigma_y\}_{y\in\Sigma_2}$ and form the ensemble $\{p(x)q(y), \rho_x \otimes \sigma_y\}_{(x,y)\in\Sigma_1\times\Sigma_2}$. Use this ensemble to show that $\chi(T \otimes S) \geqslant \chi(T) + \chi(S)$.

(2) To show the other direction, consider an ensemble $\{p(x), \rho_x\}_{x\in\Sigma}$ with $\rho_x \in D(\mathcal{H}_A \otimes \mathcal{H}_C)$. Show that for every $x \in \Sigma$ there are pure states $|b_{xy}\rangle\langle b_{xy}| \in D(\mathcal{H}_B)$ and $|c_{xy}\rangle\langle c_{xy}| \in D(\mathcal{H}_C)$ (not necessarily orthogonal!) and a probability distribution q such that

$$(T \otimes \mathrm{id}_C)(\rho_x) = \sum_y q(x, y)|b_{xy}\rangle\langle b_{xy}| \otimes |c_{xy}\rangle\langle c_{xy}|.$$

(3) For $x \in \Sigma$ consider the quantum state

$$\sigma^{(x)}_{ABD} = \sum_y q(x, y)|y\rangle\langle y|_A \otimes |b_{xy}\rangle\langle b_{xy}| \otimes S\left(|c_{xy}\rangle\langle c_{xy}|\right),$$

and compute the reduced quantum states $\sigma^{(x)}_{BD}, \sigma^{(x)}_{AB}, \sigma^{(x)}_{B}$.

(4) Use strong-subadditivity of the von-Neumann entropy to show that

$$H(\sigma^{(x)}_{BD}) \geqslant H(\sigma^{(x)}_{ABD}) - H(\sigma^{(x)}_{AB}) + H(\sigma^{(x)}_{B}).$$

(5) Use the previous inequality to show that

$$\chi\left(\{p(x), (T \otimes S)(\rho_x)\}\right) \leqslant \chi\left(\{p(x), T(\mathrm{Tr}_C\left[\rho_x\right])\}\right) + \chi\left(\{p(x)q(x, y), S(|c_{xy}\rangle\langle c_{xy}|)\}\right),$$

and conclude that $\chi(T \otimes S) \leqslant \chi(T) + \chi(S)$.