UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT 4430 — Quantum information theory (Mock exam)

Day of examination: Whenever

Examination hours: T − T+4h

This problem set consists of 8 pages.

Appendices: None

Permitted aids: none.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: The exam consists of 10 subexercises which all give maximally 10 points. To get a point for a subexercise it is expected that you give an explanation of your solution.

Problem 1

Consider the linear map $T : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ given by

$$T\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{3}x_1 + \frac{2}{3}x_4 & \frac{1}{6}x_2 \\ \frac{1}{6}x_3 & \frac{1}{3}x_4 + \frac{2}{3}x_1 \end{pmatrix}$$

1a

Show that T is a quantum channel.

Solution: Compute the Choi operator

$$C_T = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{6} \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ \frac{1}{6} & 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

This operator is positive semidefinite, since it is selfadjoint and it has eigenvalues $2/3, 1/2$ and $1/6$. Finally, T is trace-preserving since

$$\mathrm{Tr}\left[T\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right)\right] = (\frac{1}{3}x_1 + \frac{2}{3}x_4) + (\frac{1}{3}x_4 + \frac{2}{3}x_1) = x_1 + x_4 = \mathrm{Tr}\left(\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}\right).$$

1b

Compute a Kraus decomposition for T.

(Continued on page 2.)

Solution: The eigenvalue $\lambda_1 = 2/3$ has multiplicity 2 and two orthogonal eigenvectors for this eigenvalue are

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \text{ and } v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

An eigenvector for the eigenvalue $\lambda_2 = 1/2$ is

$$v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and an eigenvector for the eigenvalue $\lambda_3 = 1/6$ is

$$v_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Using the inverse vectorization and multiplying by the square root of the corresponding eigenvalue gives the Kraus operators

$$K_1 = \begin{pmatrix} 0 & \sqrt{2/3} \\ 0 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 \\ \sqrt{2/3} & 0 \end{pmatrix}, K_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, K_4 = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With those operators we have

$$T(X) = K_1 X K_1^\dagger + K_2 X K_2^\dagger + K_3 X K_3^\dagger + K_4 X K_4^\dagger.$$

1c

Compute a Stinespring dilation of T.

Solution: By "stacking" the Kraus operators we find a Stinespring isometry $V : \mathbb{C}^2 \to \mathbb{C}^8$ by

$$V = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2/3} \\ 0 & 0 \\ 0 & 0 \\ \sqrt{2/3} & 0 \\ 1/2 & 0 \\ 0 & 1/2 \\ 1/\sqrt{12} & 0 \\ 0 & -1/\sqrt{12} \end{pmatrix}.$$

We have the Stinespring dilation

$$T(X) = \mathrm{Tr}_E \left[V X V^\dagger \right],$$

where $\mathbb{C}^8 = \mathbb{C}^2 \otimes \mathbb{C}^4$ and E refers to the 4-dimensional tensor factor (in our case this is the sum of the 2×2 diagonal blocks of $VXV^\dagger$).

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1d

Is the quantum channel T entanglement breaking?

Solution: Yes, it is! Since T is a unital qubit channel it is enough to check that $\vartheta_2 \circ T$ is completely positive, where ϑ_2 denotes the transpose map in the computational basis. We have

$$C_{\vartheta_2 \circ T} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix},$$

which is positive semidefinite (we only need to check that the inner block is positive semidefinite and it is since the two principal minors are non-negative).

Problem 2

Consider a quantum channel $T : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ with Choi operator C_T and the following scenario: Alice and Bob are two scientists who each posess a qubit quantum system labelled A and B, respectively. The joint quantum state of these qubits is given by the normalized Choi operator $\rho_{AB} = C_T / \mathrm{Tr}\,[C_T]$.

2a

Assume that there is another qubit quantum system labelled A' in Alice's laboratory. Initially, this quantum system is uncorrelated with the system A and in a quantum state $\sigma_{A'}$ unknown to Alice. Compute the final quantum state τ_B of Bob's system after Alice measures her systems $A'A$ using the Bell measurement and obtained a particular outcome $(i, j) \in \{0, 1\}^2$.

Solution: The Bell measurement is the projection-valued measure $\mu : \{0, 1\}^2 \to B(\mathbb{C}^2)^+$ given by

$$\mu(i, j) = |\psi_{ij}\rangle\langle\psi_{ij}|,$$

with $|\psi_{ij}\rangle = (\mathbb{1}_2 \otimes \sigma_{ij})|\Omega_2\rangle$, where $\sigma_{00} = \mathbb{1}_2$, $\sigma_{01} = \sigma_x$, $\sigma_{10} = \sigma_y$, $\sigma_{11} = \sigma_z$. Since each quantum channel admits a Kraus decomposition, it is helpful to consider the case of a single Kraus operator $K : \mathbb{C}^2 \to \mathbb{C}^2$. In this case we have

$$C_T = 2(\mathbb{1}_A \otimes K)\omega_2(\mathbb{1}_A \otimes K^\dagger),$$

where $\omega_2 = |\Omega_2\rangle\langle\Omega_2|$ denotes the maximally entangled state. Using the computation from the standard teleportation protocol, we can compute that

$$\begin{aligned}(\langle\psi_{ij}| \otimes \mathbb{1}_B)(\sigma_{A'} \otimes C_T)(|\psi_{ij}\rangle \otimes \mathbb{1}_B) &= 2\left(\langle\psi_{ij}| \otimes K\right)(\sigma_{A'} \otimes \omega_2)\left(|\psi_{ij}\rangle \otimes K^\dagger\right) \\ &= \frac{1}{2} K\sigma_{ij}\sigma_{A'}\sigma_{ij}K^\dagger.\end{aligned}$$

(Continued on page 4.)

For a general quantum channel T, we conclude by using the Kraus decomposition and linearity that

$$(\langle\psi_{ij}| \otimes \mathbb{1}_B) \left(\sigma_{A'} \otimes \frac{C_T}{\mathrm{Tr}\,[C_T]}\right) (|\psi_{ij}\rangle \otimes \mathbb{1}_B) = \frac{1}{4} T\left(\sigma_{ij}\sigma_{A'}\sigma_{ij}\right),$$

where we used that $\mathrm{Tr}\,[C_T] = 2$ by trace-preservation. This shows that we obtain the outcome $(i,j) \in \{0,1\}^2$ with probability $1/4$ and after observing this outcome, the state of Bob's system is given by

$$\tau_B = T\left(\sigma_{ij}\sigma_{A'}\sigma_{ij}\right).$$

2b

Assume that

$$T = p_0 \mathrm{id}_2 + p_1 \mathrm{Ad}_{\sigma_x} + p_2 \mathrm{Ad}_{\sigma_y} + p_3 \mathrm{Ad}_{\sigma_3},$$

for probabilities $p_0, p_1, p_2, p_3 \in [0,1]$ summing to 1. After her measurement Alice communicates the measurement outcome $(i,j) \in \{0,1\}^2$ to Bob. What quantum channel does Bob have to apply in order to be sure that his quantum system is in the quantum state $T(\sigma_{A'})$.

Solution: By the previous exercise, Bob knows that he holds the quantum state

$$\tau_B = T\left(\sigma_{ij}\sigma_{A'}\sigma_{ij}\right).$$

after he received Alice's message. By the anticommutation relation of the Pauli matrices, we find that

$$\mathrm{Ad}_{\sigma_{kl}}(\sigma_{ij}\sigma_{A'}\sigma_{ij}) = \sigma_{kl}\sigma_{ij}\sigma_{A'}\sigma_{ij}\sigma_{kl} = \sigma_{ij}\sigma_{kl}\sigma_{A'}\sigma_{kl}\sigma_{ij} = \sigma_{ij}\mathrm{Ad}_{\sigma_{kl}}(\sigma_{A'})\sigma_{ij}.$$

Note two signs have cancelled each other in the second equation. We conclude that for the specified channel we have

$$\tau_B = T\left(\sigma_{ij}\sigma_{A'}\sigma_{ij}\right) = \sigma_{ij} T(\sigma_{A'})\sigma_{ij}.$$

Bob can therefore apply the unitary quantum channel $\mathrm{Ad}_{\sigma_{ij}}$ (as in the original teleportation protocol) to make sure that his system is in the quantum state

$$T(\sigma_{A'}).$$

Problem 3

In the following, let $\sigma_1, \sigma_2, \sigma_3 \in B(\mathbb{C}^2)$ denote the Pauli matrices. Let $P : B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ denote a linear map of the form

$$P(x) = \mathrm{Tr}\,[x]\,\frac{\mathbb{1}_2}{2} + \frac{1}{2}\lambda_1 \mathrm{Tr}\,[\sigma_1 x]\,\sigma_1 + \frac{1}{2}\lambda_2 \mathrm{Tr}\,[\sigma_2 x]\,\sigma_2,$$

(Continued on page 5.)

with $\lambda_1, \lambda_2 \in \mathbb{R}$. Assume that P is positive and show that there exists a $\lambda_3 \in [-1, 1]$ such that the map

$$T = P + \frac{1}{2}\lambda_3 \mathrm{Tr}\left[\sigma_3 x\right] \sigma_3,$$

is a quantum channel. Determine for which λ_1 and λ_2, the quantum channel T can be entanglement breaking.

Solution: Each map of the stated form is unital and trace-preserving. Since P is assumed to be positive, we know that $\lambda_1, \lambda_2 \in [-1, 1]$. The parameters $(\lambda_1, \lambda_2, \lambda_3)$ defining unital qubit channels form a tetrahedron inside the cube $[-1, 1]^3$. Now, envision the point $(\lambda_1, \lambda_2, 0)$ in the x-y-plane. For each such point we can go up or down in the z-direction and hit the tetrahedron, or ,to say it in more high-level terms, the projection of the tetrahedron to the x-y-plane is the entire square $[-1, 1]^2$. In other words, there always exists a $\lambda_3 \in [-1, 1]$ such that the stated T is a quantum channel.
For the second question, recall that the entanglement breaking unital qubit channels form an octahedron. Since projecting this octahedron to the x-y-plane coincides with the intersection of the octahedron with the x-y-plane we see that we can only obtain an entanglement breaking qubit channel if the map P was an entanglement breaking qubit channel to begin with.

Problem 4

Imagine you are a quantum telecommunication engineer. You have a device to produce a qubit in either of the states $\rho_0, \rho_1 \in D(\mathbb{C}^2)$ given by

$$\rho_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho_1 = |+\rangle\langle +| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and send it to a distant location.

4a

Find a POVM $\mu : \{0, 1\} \to B(\mathbb{C}^2)$ such that the classical channel $N_\mu : \{0, 1\} \to \mathcal{P}(\{0, 1\})$ with $N_\mu(x|y) = \mathrm{Tr}\left[\mu(x)\rho_y\right]$ is binary symmetric. Compute the capacity of the classical channel N_μ .

Solution: Observe that

$$\rho_0 = \frac{1}{2}\left(\mathbb{1}_2 + \sigma_z\right),$$

and

$$\rho_1 = \frac{1}{2}\left(\mathbb{1}_2 + \sigma_x\right),$$

(Continued on page 6.)

correspond to the vectors

$$v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ and } v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

on the Bloch sphere. From the way the spinor map transforms angles, we can try to find two orthogonal pure qubit states (which will form a projection-valued measure) such that their overlaps with the vectors v_0 and v_1 are symmetric. From the Bloch sphere representation, we might get the idea to use

$$w_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \text{ and } w_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

which lead to overlaps

$$\begin{aligned}
\langle w_0, v_0 \rangle &= \frac{1}{\sqrt{2}}, \\
\langle w_1, v_0 \rangle &= -\frac{1}{\sqrt{2}}, \\
\langle w_0, v_1 \rangle &= -\frac{1}{\sqrt{2}}, \\
\langle w_1, v_1 \rangle &= \frac{1}{\sqrt{2}}.
\end{aligned}$$

Now, we can transform the vectors w_0, w_1 to pure states giving rise to the projection-valued measure $\mu : \{0, 1\} \to B(\mathbb{C}^2)$ with

$$\mu(0) = \frac{1}{2} \left(\mathbb{1}_2 + \frac{1}{\sqrt{2}} (\sigma_z - \sigma_x) \right)$$

and

$$\mu(1) = \frac{1}{2} \left(\mathbb{1}_2 + \frac{1}{\sqrt{2}} (-\sigma_z + \sigma_x) \right).$$

With this we find that

$$\begin{aligned}
\operatorname{Tr}\left[\mu(0)\rho_0\right] &= \frac{1}{2} + \frac{1}{2\sqrt{2}}, \\
\operatorname{Tr}\left[\mu(1)\rho_0\right] &= \frac{1}{2} - \frac{1}{2\sqrt{2}}, \\
\operatorname{Tr}\left[\mu(0)\rho_1\right] &= \frac{1}{2} - \frac{1}{2\sqrt{2}}, \\
\operatorname{Tr}\left[\mu(1)\rho_1\right] &= \frac{1}{2} + \frac{1}{2\sqrt{2}}.
\end{aligned}$$

With this the channel $N_\mu : \{0, 1\} \to \mathcal{P}\left(\{0, 1\}\right)$ is binary symmetric with flipping probability

$$p = \frac{1}{2} - \frac{1}{2\sqrt{2}} = 0.1464\ldots.$$

(Continued on page 7.)

4b

Show that you can send classical information at a rate at least 0.3991 bits per use of your device.

Solution: The capacity of the binary symmetric channel from the previous exercise is given by

$$C(N_\mu) = 1 - h_2(p) = 1 - h_2(1-p) = 1 - 0.6009\ldots \approx 0.3991.$$

4c

Argue that we can send classical information at a rate of at least 0.6 bits per use of the device, if we allow for global measurements at the receiving end.

Solution: We know that the Holevo-information of any ensemble $\{p_x, \rho_x\}_{x\in\{0,1\}}$ is an achievable rate for classical communication if we allow for global measurements. We can compute

$$\chi\left(\{p_x, \rho_x\}_{x\in\{0,1\}}\right) = H(\sum_x p_x\rho_x) - \sum_x p_x H(\rho_x) = H(\sum_x p_x\rho_x),$$

since ρ_0 and ρ_1 are pure. Choosing $p_0 = p_1 = 1/2$ we find that

$$H(\rho_0/2 + \rho_1/2) = H\left(\begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}\right),$$

is achievable. Since the spectrum of the state in the entropy is also given by $1/2 \pm 1/2\sqrt{2}$ we find that $H(p) = 0.6009$ is an achievable rate.

(Continued on page 8.)

Matlab-utskrift.

```
>> A=[3/4 1/4
1/4 1/4]

A =

     3/4      1/4
     1/4      1/4


>> eig(A)

ans =

    0.8536
    0.1464

>> x=1/2 + 1/(2*sqrt(2))

x =

    0.8536

>> -x*log2(x)-(1-x)*log2(1-x)

ans =

    0.6009
```