OBLIGATORY PROBLEMS IN MAT 4450

Deadline May 8, 2015, 2:30 p.m.

Solutions must be delivered in the special box ("obligkassa") on the 7th floor in the Niels Henrik Abel building.

Remark. You must justify your answers by including an adequate amount of details. Each question is weighted as indicated. To pass the assignment you will need a score of at least 50 points.

Note: You must provide details for your answers. The individual questions have different weights, as indicated. To pass the assignment you will need a score of at least 50 points.

Problem 1. In this problem $\mathbb{R}$ is the real line with its usual topology given by open intervals. Consider the following (real) vector spaces:

$$C_b(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous and } \sup_{x\in\mathbb{R}} |f(x)| < \infty\};$$
$$C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous and } \forall\varepsilon > 0, \{x \mid |f(x)| \geq \varepsilon\} \text{ is compact}\};$$
$$C_c(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous and } \overline{\{x \mid f(x) \neq 0\}} \text{ is compact}\}.$$

Thus $C_b(\mathbb{R})$ is the space of bounded continuous functions, $C_0(\mathbb{R})$ is the space of continuous functions which vanish at infinity, and $C_c(\mathbb{R})$ is the space of continuous functions with compact support (recall that $\overline{\{x \mid f(x) \neq 0\}}$ is the support of f). It is known that $C_c(\mathbb{R}) \subset C_0(\mathbb{R}) \subset C_b(\mathbb{R})$. Let $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$ for $f \in C_b(\mathbb{R})$ be the uniform norm. It is assumed known that $C_b(\mathbb{R})$ is a Banach space with $\|\cdot\|_\infty$.

(a) (10 points) Define for each $g \in C_0(\mathbb{R})$ a map $m_g : C_b(\mathbb{R}) \to \mathbb{R}$ by $m_g(f) = \|fg\|_\infty$ for all $f \in C_b(\mathbb{R})$. Show that $\mathcal{F} = \{m_g \mid g \in C_0(\mathbb{R})\}$ is a separating family of seminorms on $C_b(\mathbb{R})$.

(b) (10 points) Give an example of a sequence $(k_n)_{n\geq 1}$ in the space $C_c(\mathbb{R})$ such that $\|g - gk_n\|_\infty \to 0$ for every $g \in C_0(\mathbb{R})$. Conclude that $C_c(\mathbb{R})$ is dense in $C_b(\mathbb{R})$ with respect to the seminorm topology given by $\mathcal{F}$.

(c) (10 points) Show that there is a sequence $(f_n)_{n\geq 1}$ in $C_c(\mathbb{R})$ such that $\|f_n g\|_\infty \to 0$ for each $g \in C_c(\mathbb{R})$ but it does not hold that $f_n \to 0$ in the seminorm topology on $C_b(\mathbb{R})$ from $\mathcal{F}$. (Hint: you can use $f_n(x) = nh(x+n)$ for $n \geq 1$, where $h(x) = \max\{1 - |x|, 0\}$ for $x \in R$.)

(d) (5 points) For each $f \in C_b(\mathbb{R})$ define $M_f : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ by $M_f(g) = fg$ for $g \in C_0(\mathbb{R})$. Show that $f \to M_f$ is a norm preserving linear map from $C_b(\mathbb{R})$ into $B(C_0(\mathbb{R}))$ (the last one endowed with the usual operator norm).

(e) (5 points) Use (d) to conclude that if a sequence (f_n) in $C_b(\mathbb{R})$ is such that $f_n \to 0$ in the seminorm topology from $\mathcal{F}$, then $(\|f_n\|)_n$ must be a bounded sequence.

Problem 2. Let H be an infinite-dimensional Hilbert space.

(a) (5 points) Let $S \in B(H)$ be such that $S = S^*$ (i.e. S is self-adjoint), and suppose that $S = U|S|$ is the polar decomposition of S. Show that $U = U^*$ and that U commutes with $|S|$.

(b) (5 points) With notation as in (a), show that $-I \leq U \leq I$ and $-|S| \leq S \leq |S|$.

(c) (5 points) Given $S \in B(H)$ such that $S = S^*$, show that $-\|S\|I \leq S \leq \|S\|I$. Deduce from this that a self-adjoint operator in $B(H)$ is a difference of positive operators in $B(H)$.

(d) (5 points) Show that every T in $B(H)$ is a linear combination of at most four positive operators. Hint: Use $T_1 = (T + T^*)/2$ and $T_2 = (T - T^*)/(2i)$.

Problem 3. (20 points) (*Right Fredholm operators.*) Prove that the following conditions on T in $B(H)$ are equivalent:

(1) There is a unique S in $B(H)$ such that TS is the projection onto $(\ker T^*)^\perp$ and has finite co-rank.
(2) There is $S \in B(H)$ such that $TS - I$ is compact.
(3) The image of T in the quotient normed space $B(H)/B_0(H)$ is an invertible operator.
(4) The subspace $T(H)$ is closed and has finite co-dimension.
(5) The operator $(TT^*)^{1/2}$ is Fredholm.