Banach-Alaoglu theorem

Let X be a vector space, Y a space of linear functionals on X separating points (that is, $\forall x \in X\setminus\{0\}\ \exists f \in Y$ s.t. $f(x) \neq 0$). Then $(X, \sigma(X,Y))$ can be identified with a subset of $\prod_{f \in Y\setminus\{0\}} \mathbb{C}$. Namely, we map $x \in X$ into $(f(x))_{f \in Y\setminus\{0\}}$.

This observation leads to the following fundamental result:

Theorem (Banach-Alaoglu)

Let X be a normed space. Then the closed unit ball in X^* is w^*-compact.

Proof

Consider the closed unit balls $\overline{B}_X$, $\overline{B}_{X^*}$, $\overline{\mathbb{D}}$ in X, X^*, $\mathbb{C}$, respectively. Then, similarly to the above observation, $(\overline{B}_{X^*}, w^*)$ can be identified with a closed subset of

$$\prod_{x \in \overline{B}_X} \overline{\mathbb{D}} \text{ via the map } f \mapsto (f(x))_{x \in \overline{B}_X}.$$

As $\prod_{x \in \overline{B}_X} \overline{\mathbb{D}}$ is compact by Tychonoff's theorem, we conclude that $(\overline{B}_{X^*}, w^*)$ is compact. ☒

As an application let us show the existence of <u>Banach limits</u>.

Def. A <u>Banach limit</u> is a linear functional ℓ on $\ell^\infty = \ell^\infty(\mathbb{N})$ s.t.

(i) $\ell \geq 0$; that is, if $x_n \geq 0$ for all $n \geq 1$, then $\ell((x_n)_{n=1}^\infty) \geq 0$;

(ii) if $\lim_{n\to\infty} x_n$ exists, then $\ell((x_n)_{n=1}^\infty) = \lim_{n\to\infty} x_n$;

(iii) $\ell((x_n)_{n=1}^\infty) = \ell((x_{n+1})_{n=1}^\infty)$ $\forall (x_n)_{n=1}^\infty \in \ell^\infty$.

It can be shown that (i) and (ii) imply $\|\ell\| = 1$. To show that Banach limits exist, consider the functionals $\ell_n \in (\ell^\infty)^*$ defined by

$$\ell_n((x_m)_{m=1}^\infty) = \frac{1}{n}\sum_{m=1}^n x_m .$$

Then $(\ell_n)_{n=1}^\infty$ is a sequence in the unit ball of $(\ell^\infty)^*$. By the Banach-Alaoglu theorem it must have a w^*-cluster point ℓ. Since $\ell(x)$ is a cluster point of $(\ell_n(x))_{n=1}^\infty$, properties (i), (ii) should be clear.

In order to check (iii), consider the operator $S: \ell^\infty \to \ell^\infty$, $S((x_n)_{n=1}^\infty) = (x_{n+1})_{n=1}^\infty$. We have to check that $\ell = \ell \circ S$. Since $\ell - \ell\circ S$ is a w^*-cluster point of $(\ell_n)_{n=1}^\infty$, it suffices to show that $\ell_n - \ell_n \circ S \xrightarrow[n\to\infty]{w^*} 0$. But this is true even in norm!

$(\ell_n - \ell_n\circ S)((x_n)_{n=1}^\infty) = \frac{1}{n}(x_1 - x_{n+1})$, so $\|\ell_n - \ell_n \circ S\| \leq \frac{2}{n} \xrightarrow[n\to\infty]{} 0$

Hence ℓ is indeed a Banach limit.

<u>Example</u>
Consider $x = (1,0,1,0,\dots)$ and let ℓ be a Banach limit.
Then $\ell(x) = \frac{1}{2}\ell(x + S(x)) = \frac{1}{2}\ell(1,1,\dots) = \frac{1}{2}$.