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Introduction

Topics

- Locally convex topological vector space
  - convex sets
  - pointwise convergence
- Spectral theory of operators on Hilbert space
  - normal bounded operators
  - self adjoint unbounded operators

Locally convex top. vec. sp.

$\mathfrak{X}, \mathcal{Y}, \mathfrak{Z}, \ldots$ vec. sp. over $\mathbb{C}$ (or $\mathbb{R}$)

with topology compatible with $u+v$, αu

(and <u>convex</u> combinations)

Example : Banach space $\|u\| \in [0, \infty)$ makes sense

1) X compact Hausdorff top. sp.

$\rightsquigarrow C(X) = \{ f : X \to \mathbb{C} \text{ cont.} \}$, $\|f\| = \max\limits_{x \in X} |f(x)|$

2) (X, μ) measure sp. $1 \leq p < \infty$

$L^p(X, \mu)$ "=" $\{ f : X \to \mathbb{C}$ measurable

$\underbrace{\int_X |f(x)|^p \, d\mu(x)}_{\text{"}(\|f\|_p)^p} < \infty \}$

3) Space of maps $\mathcal{L}(\mathfrak{X}, \mathcal{Y})$

$\|T\| = \sup \|Tu\| \; ; \; \|u\| \leq 1.$

in particular $\mathcal{Y} = \mathbb{C} \rightsquigarrow \underbrace{\mathfrak{X}^* = \mathcal{L}(\mathfrak{X}, \mathbb{C})}_{\text{dual space}}$

$L^p(X, \mu)^* = L^q(X, \mu)$ for $\frac{1}{p} + \frac{1}{q} = 1$

Why go beyond Banach spaces ?

Pointwise conv. top. on $\mathcal{L}(\mathfrak{X}, \mathcal{Y})$

cont.) $(T_n)_n \to T$ in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$:

$\forall u \in \mathcal{X}$ $T_n u \to Tu$ in $\mathcal{Y}$.

Ex. $\mathcal{X} = \mathcal{Y} = \ell_2 \mathbb{N} = \{(a_n)_{n=1}^{\infty} : \sum |a_n|^2 < \infty\}$

$T_1 (a_n)_n = (b_n)_n$, $b_n = a_{n+1}$

$T_n = T_1^n$. $\rightsquigarrow$ $T_n u \to 0$ for $u \in \ell_2 \mathbb{N}$.

But $\|T_n\| = 1$.

Convex set : $K \subset \mathcal{X}$ s.t. $u, v \in K$, $0 \leq t \leq 1$

$\Rightarrow tu + (1-t)v \in K$

Motivating problem : invariant measures

X : cpt top. sp. $T : X \to X$ homeo

¿ Why $\exists$ T-invariant regular Borel probability measure? i.e.

$\exists \mu$ $\underline{\int f(Tx)\, d\mu(x)} = \int f(x)\, d\mu(x)$

Rough idea , $\overset{\shortparallel}{\int f \, d T_{\#}\mu}$.

Pick <u>any</u> prob. measure μ_1

$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} T^k{}_{\#} \mu_1$

$\rightsquigarrow (\mu_n)_{n=1}^{\infty}$ is asymptotically invariant

$\mu_n - T_{\#}\mu_n = \frac{1}{n}(\mu_1 - T^n{}_{\#}\mu_1)$. "$\to 0$"

$\Rightarrow$ "$\mu_\infty = \lim_{n\to\infty} \mu_n$" would be T-invar.

¿ How do we make sense of the limit?

- Spectral theory of operators on Hilbert spaces

$X \in M_n(\mathbb{C})$ is <u>normal</u> if $X^* = \bar{X}^t$ satisfies

$XX^* = X^*X$

Normal $\rightsquigarrow \exists$ unitary U s.t. UXU^* diag.

H : Hilbert space Hermitian inn. prod (u, v)

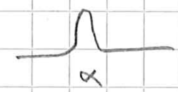
$\rightsquigarrow$ T^* is characterized by $(Tu, v) = (u, T^*v)$

"Normal operators have good theory of eigenvalues". (or unbounded selfadjoint ops)

But : we cannot always have eigenvectors

$\mathcal{H} = L^2([0,1], \mu_{Leb})$, $(Tu)(t) = t\,u(t)$

$\rightsquigarrow$ $T = T^*$ but $Tu = \alpha u$ would imply "$u = \delta_\alpha$"

Still : f  is almost an eigenvector.

Motivating problem : Dirichlet's prob. (a variant)

$$-\frac{d^2 u}{dt^2}(t) + V(t)\,u(t) = \alpha\, u(t)$$

$u(0) = u(1) = 0$ (boundary condition)

for $V(t)$: cont. func. for $0 \leq t \leq 1$

α : (complex) scalar

¿ which α admit nontrivial solution for $u(t)$?

$\rightsquigarrow$ When α belongs to a discrete subset of $\mathbb{R}$

Ex. $V(t) = 0$ $\rightsquigarrow$ $\sin 2\pi k t$ are solutions for $\alpha = (2\pi k)^2$

This is an eigenvalue problem of

$$T : u \longmapsto -\frac{d^2 u}{dt^2} + V \cdot u.$$

on $\{u \in C^2([0,1]),\ u(0) = u(1) = 0\} \subset L^2([0,1], \mu_{Leb})$