Summary

- Top. Vec. Sp.
  - loc. conv. TVS
- Weak top.
  - Weak top. from pairing.

$\mathcal{X}$: <u>TVS</u> over $\mathbb{C}$ means

$\mathcal{X}$ vec sp. ($u+v$, αu make sense)

$\mathcal{X}\times\mathcal{X}\to\mathcal{X}$, $(u,v)\mapsto u+v$

$\mathbb{C}\times\mathcal{X}\to\mathcal{X}$, $(\alpha,v)\mapsto\alpha v$ cont.

Rem. Similar for TVS over $\mathbb{R}$ (or any top. field)

TVS is <u>loc. convex</u> if

$0\in\mathcal{X}$ (or $\forall u\in\mathcal{X}$) has enough conv. neighs, $\forall U\in\mathcal{X}$ open, $0\in U$

$\Rightarrow \exists K$ conv. V open $0\in V\subset K\subset U$.

Rem. Loc. conv $\Rightarrow$ enough balanced[1] & absorbing[2] conv. neigh of 0.

1) $\alpha\in\mathbb{C}$, $|\alpha|\le 1\Rightarrow \alpha K\subset K$.

2) $\bigcup_{\alpha\in\mathbb{C}}\alpha K=\mathcal{X}$.

K conv. neigh of 0 $\Rightarrow \tilde{K}=\overline{\mathrm{Conv}}\left(\bigcup_{|\alpha|\le 1}\alpha K\right)$

is still conv neigh. of 0.

abs: $u\in\mathcal{X}\Rightarrow \alpha u\in K$ for small enough α

$\Rightarrow \alpha^{-1}\tilde{K}\ni u$

bal: "from const"

Rem. Enough to specify neigh. of 0.

A neigh of 0 $\Rightarrow u+A$ neigh of u.

Ex. 1) Banach spaces.

$B_r(0) = \{ u \in \mathcal{X} : \|u\| < r \}$ is

conv. (bal, abs)

2) $C^\infty([0,1]) = \{ f(t) : 0 \le t \le 1 , \ \partial^k f = \frac{d^k f}{dt^k}(t)$ exist for all $k \}$

Neigh of 0 :

$A_{k,r} = \{ f : |(\partial^k f)(t)| < r \quad \forall t \}$

(So : $f_1, f_2, \dots \to f$ in $C^\infty([0,1])$

iff $\forall k$ $\partial^k f_n \to \partial^k f$ uniformly

(Top of $C^\infty([0,1])$ is def'd by

<u>seminorms</u> $\|f\|_{(k)} = \|\partial^k f\|_{C([0,1])}$)

◦ Weak top.

$\mathcal{X}$ (loc. conv) TVS. its <u>dual space</u> is

$\mathcal{X}^* = \{ \phi : \mathcal{X} \to \mathbb{C}$ lin. cont. $\}$

(also write $\mathcal{X}'$).

The <u>wk top</u> on $\mathcal{X}$: base of opens

$\mathcal{O}_{\phi_1, \dots, \phi_k, A_1, \dots, A_k} = \{ u : \phi_n(u) \in A_n \}$

for $\phi_n \in \mathcal{X}^*$, $A_n \subset \mathbb{C}$ open.

Rem: General open : union of these

$\mathcal{O}_{\phi_1 \dots \phi_k, A_1, \dots A_k} = \bigcap_{n=1}^{k} \mathcal{O}_{\phi_n, A_n}$.

So $u_1, u_2, \dots \to u$ in the wk. top.

if $\forall \phi \in \mathcal{X}^*$ $\phi(u_n) \to \phi(u)$.

Rem Why call this "weak" ?

~~$\mathcal{X}$ Banach sp $\rightsquigarrow$ $\mathcal{X}^*$ is also Ban.~~

~~$\|\phi\| = \sup |\phi(u)| : \|u\| \le 1$.~~

Strong top : orig top. on $\mathcal{X}$.

Ex. Hilb. sp. $\mathcal{H}$

(conj. lin.) ident. $\mathcal{H} \tilde{\to} \mathcal{H}^*$, $u \mapsto (-, u)$

$\mathcal{H} = \ell_2 \mathbb{N} = \{ (a_k)_{k=1}^{\infty} : \sum |a_k|^2 < \infty \}$

$u_n = (0, \dots, 0, \underset{n\text{-th}}{1}, 0, \dots)$

$\|u_n\| = 1$ for $\forall n$, but

$v = (a_k)_k \rightsquigarrow (u_n, v) = \bar{a}_n \to 0 \quad (n \to \infty)$

so $u_n \to 0$ in wk top.

- Generaliz. to pairing.

$\mathcal{X}$ (loc conv.) TVS, $\mathcal{Y} \subset \mathcal{X}^*$ subsp

$\sigma(\mathcal{X}, \mathcal{Y})$ - topology on $\mathcal{X}$:

base opens : $\mathcal{O}_{\phi_1, \dots, \phi_k, A_1, \dots, A_k}$ as before

$(= \{ u : \forall n \ \ \phi_n(u) \in A_n \})$ but

$\phi_n \in \mathcal{Y}$, $A_n \subset \mathbb{C}$ open.

Important Ex: 1. $\mathcal{Y} = \mathcal{X}^*$ wk top.

2. $\mathcal{X} = \mathcal{Y}^*$, reg. $\mathcal{Y}$ as subset of $\mathcal{Y}^{**}$

by $u(\phi) = \phi(u)$ $u \in \mathcal{Y}$, $\phi \in \mathcal{X}$.

$\rightsquigarrow \sigma(\mathcal{X}, \mathcal{Y})$ is called weak*-top. on $\mathcal{X}$.

Caution : no uniqueness of $\mathcal{Y}$ (predual) in gen.

But von Neumann algs are OK.

Th'm. In the above setting $\mathcal{Y} = \sigma(\mathcal{X}, \mathcal{Y})$-cont lin. funcs.

Pf. $\phi \in \mathcal{Y}$ is $\sigma(\mathcal{X}, \mathcal{Y})$-cont : back to def.

Let ϕ be a $\sigma(\mathcal{X}, \mathcal{Y})$-cont functional

Step 1 : $\exists\, \phi_1, \dots, \phi_k, r_1, \dots, r_k : \bigcap \Theta_{\phi_n, \{|z|<r_n\}} \subset \phi^{-1}(|z|<1)$

$\phi^{-1}(\{z \in \mathbb{C} : |z| < 1\})$ is $\sigma(\mathcal{X}, \mathcal{Y})$-opn.

$$\Theta_{\phi_1, \dots, \phi_n, A_1, \dots, A_n} \subset \phi^{-1}(\{|z|<1\})$$

for some $\phi_n \in \mathcal{Y}$, $A_n \subset \mathbb{C}$ opn.

$0 \in \phi^{-1}(\{z\} < 1\}) \Rightarrow 0 \in A_n \Rightarrow$ WMA

$A_n = \{|z| < r_n\}$ for some r_n.

Step 2 ϕ factors through $\mathcal{X} \to \mathbb{C}^k$

$u \mapsto (\phi_1(u), \dots, \phi_k(u))$

$\because$ $\phi_n(u) = 0$ for all $n \Rightarrow \phi(u) = 0$.

Step 3 ϕ is a lin. comb. of $(\phi_n)_{n=1}^k$

from Step 2.