MAT 4450 (V2019) Lecture 3 (for Makoto Y.) 21.01.2019

Recall defs: - X TVS over $\mathbb{C}$

- X TVS loc. convex if $\forall U \subseteq X$ open with $0 \in U$, there are K convex $\subseteq X$, $\mathcal{O} \subseteq X$ open s.t. $0 \in \mathcal{O} \subseteq K \subseteq U$

Fact / non-trivial: If X is loc conv TVS there are "enough" balanced absorbing conv. neighbourhoods of 0.

1) balanced: $\alpha \in \mathbb{C}, |\alpha| \leq 1 \Rightarrow \alpha K \subseteq K$

2) absorbing: $\bigcup_{\alpha \in \mathbb{C}} \alpha K = X$.

Fact: If K convex neighbhd of 0 then $\tilde{K} \overset{\text{not.}}{=} \text{conv}\left(\bigcup_{|\alpha|\leq 1} \alpha K\right)$ is still conv. neighbhd of 0.

It is balanced by construction and absorbing since for $x \in X$, $\alpha x \in K$ for small enough α, thus $x \in \alpha^{-1}\tilde{K}$.

Weak topology: Given X (loc. conv. — not needed for def.) TVS its dual space is $X^* = \{\phi: X \to \mathbb{C} \mid \phi \text{ lin, cont.}\}$

[We'll show: when X is loc. conv, X^* is large (separates the points of X), in particular it is non-trivial.]

The weak top on X has, by def, a basis of open sets

$$\mathcal{O}_{\phi_1,\dots,\phi_k,A_1,\dots,A_k} = \{u \in X \mid \phi_j(u) \in A_j\} \quad \text{for } \phi_j \in X^*, A_j \subseteq \mathbb{C} \text{ open}$$

Then a general open set for the w-top will be a union of these sets.

where $\mathcal{O}_{\phi_1,\dots,\phi_k,A_1,\dots,A_k} = \bigcap_{j=1}^{n} \underbrace{\mathcal{O}_{\phi_j,A_j}}_{\text{subbasis}}$.

Seq. conv in w-top: $\{u_n\} \xrightarrow{w} u \iff \forall \phi \in X^*, \phi(u_n) \to \phi(u)$.

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(2)

Lemma Let X be a TVS and C a convex open neighbourhood of 0. Let $m: X \to \mathbb{R}_+$ be given by $m(x) = \inf\{s>0 \mid s^{-1}x \in C\}$. Then

(a) $m(x+y) \leq m(x) + m(y)$, $\forall x, y \in X$,

(b) $m(\alpha x) = \alpha m(x)$, $\forall x \in X$, $\forall \alpha \geq 0$,

(c) $C = \{x \in X \mid m(x) < 1\}$.

We call a functional on X with (a)-(b) a Minkowski functional.

Proof: Step 1: $m(x) < \infty$, $\forall x \in X$.

Given $x \in X$, form $x_n = \frac{1}{n}x$ for all $n \geq 1$. Then $x_n \to 0 \in C$ open, so $\exists n_0 \in \mathbb{N}$ s.t. $n \geq n_0 \Rightarrow \frac{1}{n}x \in C$. Thus, $\{n \mid n \geq n_0\} \subseteq \{s>0 \mid s^{-1}x \in C\}$, so at least $m(x) < \infty$.

Step 2: Let $t \geq 0$. Want $m(tx) = t\,m(x)$ for $x \in X$. Fix it. Prove 2 ineq. for $t \neq 0$.

At $t=0$, need $m(0 \cdot x) = m(0) = 0$. But any $s>0$ is in $\{s>0 \mid s^{-1}0 \in C\}$ so $m(0) \leq 0 = \inf\{s>0 \mid s^{-1}0 \in C\}$ and since $m(x) \geq 0$ always, get claim.

For $t>0$, let $s>0$ with $s^{-1}x \in C$, then $(s^{-1}t^{-1})\,tx \in C$ so $m(tx) \leq st$. Since s arbitrary, $m(tx) \leq m(x) \cdot t$, $\forall t>0$, x arbitrary.

Thus also $m\left(\frac{1}{t} \cdot (tx)\right) \leq m(tx) \cdot \frac{1}{t}$ i.e. $m(tx) \geq t\,m(x)$ giving $=$.

Step 3 Let $x, y \in X$. We'll show (a). Let s, t s.t. $s^{-1}x \in C$, $t^{-1}y \in C$. Then $(s+t)^{-1}(x+y) = \frac{s}{s+t} \cdot \frac{1}{s}x + \frac{t}{s+t} \cdot \frac{t}{y} \in C$ by convexity, so $m(x+y) \leq s+t$. Take inf over s, then over t to get the inequality (subadditivity)!

Step 4 Prove (c). Let $x \in C$. Since C open and $(1+\frac{1}{n})x \to x$, there is $n_0 \in \mathbb{N}$ s.t. $(1+\frac{1}{n})x \in C$ for all $n \geq n_0$, thus $m(x) \leq \frac{1}{1+1/n_0} < 1$.

Let $m(x) < 1$. Then there is $0<s<1$ st. $s^{-1}x \in C$. Thus,

$x = (1-s) \cdot 0 + s(s^{-1}x) \in C$ by convexity.