The Hahn-Banach separation theorem

Let X be a TVS over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) and suppose A, B are disjoint, nonempty, convex subsets of X. If A is open, there exists φ in $X^* = \{\varphi : X \to \mathbb{F} \mid \varphi \text{ linear, continuous}\}$ and $t \in \mathbb{R}$ s.t.

$$\operatorname{Re}\varphi(x) < t \leq \operatorname{Re}\varphi(y) \quad \forall x \in A \text{ and } \forall y \in B.$$

Proof: Case $\mathbb{F} = \mathbb{R}$ first

Take any $x_0 \in A$, $y_0 \in B$ and let $z = y_0 - x_0$. Define $C = A - B + z$.

Then $C = \bigcup_{y \in B} (A - y + z)$, where each $A - y + z$ open, so C is open.

Claim: C is convex. Let $x_1, x_2 \in A$, $y_1, y_2 \in B$, $0 < \lambda < 1$.

$$\lambda(x_1 - y_1 + z) + (1-\lambda)(x_2 - y_2 + z) = \underbrace{\underbrace{(\lambda x_1 + (1-\lambda)x_2)}_{\in A} - \underbrace{(\lambda y_1 + (1-\lambda)y_2)}_{\in B} + z}_{\in C}$$

by convexity

Claim: C is a neighbhd of 0

Clearly $0 = x_0 - y_0 + z \in C$ and C open.

By previous lemma, there is a Minkowski functional on X

$$m(w) = \inf\{s > 0 \mid s^{-1}w \in C\} \text{ s.t. } C = \{w \in X \mid m(w) < 1\}.$$

Note: $z \notin C$ (else $0 = x - y$ for some $x \in A$, $y \in B$, contradicting $A \cap B = \emptyset$).

Thus, $m(z) \geq 1$. Define

$$\varphi_0 : \mathbb{R}z \to \mathbb{R}, \quad \varphi_0(sz) = s.$$

For $s \geq 0$, $\varphi_0(sz) = s \leq s\, m(z) = m(sz)$ by properties of m, and for $s < 0$, $\varphi_0(sz) = s < 0 \leq m(sz)$.

Thus, $\varphi_0 \leq m|_{\mathbb{R}z}$. By the $\mathbb{R}$-version of the Hahn-Banach extension thm, $\exists$ extension $\varphi : X \to \mathbb{R}$, linear, s.t. $\varphi \leq m$.

We'll prove $\varphi \in X^*$, i.e. φ continuous. By linearity, enough at $x=0$.

Note that $\varphi(w) \leq m(w) < 1$ for all $w \in C$.

Let $\varepsilon > 0$; $\varepsilon C = \{\varepsilon x - \varepsilon y + \varepsilon z \mid x \in A, y \in B\}$ is an (open) neighbhd of 0.

$\varphi(\varepsilon w) = \varepsilon \varphi(w) < \varepsilon$ for all $\varepsilon w \in \varepsilon C$ and similarly

$\varphi(-\varepsilon w) = -\varepsilon \varphi(w) > -\varepsilon$ for $w \in C$ so $|\varphi(w')| < \varepsilon$ $\forall w' \in \varepsilon C \cap (-\varepsilon C)$

(open) neighbhd of 0.

Continuity of φ follows.

Separation of A and B

$\forall x, y$, $\varphi(x-y+z) \leq m(\underbrace{x-y+z}_{\in C}) < 1 = \varphi_0(z) = \varphi(z)$ so $\varphi(x) < \varphi(y)$

for all $x \in A$ and $y \in B$. Thus $\varphi(A)$ and $\varphi(B)$ are disjoint subsets of $\mathbb{R}$. By convexity of A, B and lin. of φ, $\varphi(A)$ and $\varphi(B)$ are convex subsets of $\mathbb{R}$, hence they must be intervals.

Claim: $\varphi(A)$ is open.

Let $\varphi(x) \in \varphi(A)$ for $x \in A$. Will find $\delta > 0$ s.t. $|\varphi(x') - \varphi(x)| < \delta$, $\forall x' \in A$.

Continuity of $f_{x,z} : \mathbb{R} \to X$, $f_{x,z}(s) = x + sz$ at 0 implies,

since $f_{x,z}(0) = x \in A$ open, $\exists \delta > 0$ s.t. $|s| < \delta \Rightarrow x + sz \in A$.

$\Rightarrow |\varphi(x+sz) - \varphi(x)| = |\varphi(sz)| = |s| < \delta$, $x + sz \in A$, as wanted.

Now let $t = \sup\{\varphi(x) \mid x \in A\}$ i.e. the right end-point of A. We get $\varphi(x) < t \leq \varphi(y)$ $\forall x \in A, y \in B$, as claimed.

Case $\mathbb{F} = \mathbb{C}$; find $\psi : X \to \mathbb{R}$ s.t. $\psi(x) < t \leq \psi(y)$ $\forall x \in A, y \in B$.

Put $\varphi(w) = \psi(w) - i\psi(iw)$ for $w \in X$. Get φ a $\mathbb{C}$-linear map with $\operatorname{Re}\varphi = \psi$ satisfying the claim of thm.

enough $\varphi(iw) = i\varphi(w)$, $\forall w \in X$