MAT4450 (V2019) Lecture 4 (for Makoto)

Application of Hahn-Banach separation theorem

First recall definition of w^*-topology:

Spse $\mathcal{X}$ (loc. conv.) TVS, $\mathcal{Y} \subseteq \mathcal{X}^*$ subspace of cont. lin functionals.

Define the $\sigma(\mathcal{X}, \mathcal{Y})$-topology on $\mathcal{X}$ by declaring a base of open sets:

$$\left\{ \underset{\{u \in \mathcal{X} \mid \phi_j(u) \in A_j,\ \forall j = \overline{1,k}\}}{\underset{\shortparallel}{\mathcal{O}_{\phi_1,\dots,\phi_k, A_1,\dots,A_k}}} \;\middle|\; \phi_1,\dots,\phi_k \in \mathcal{Y},\ \underset{\text{open}}{A_1,\dots,A_k \subseteq \mathbb{C}} \right\}$$

Important example 1: $\mathcal{Y} = \mathcal{X}^*$ which gives the w-top on $\mathcal{X}$ (from $\mathcal{X}^*$).

Important example 2: $\mathcal{X} = \mathcal{Y}^*$ for some $\mathcal{Y}$ l.c. TVS

$$\mathcal{Y} \subseteq \mathcal{X}^* = \mathcal{Y}^{**}$$
$$u \longmapsto j(u), \quad j(u)(\phi) = \phi(u), \ \phi \in \mathcal{Y}^*.$$

The ensuing top $\sigma(\mathcal{Y}^*, \mathcal{Y}^{**})$ is the w^*-top on $\mathcal{X}$.

Theorem In the above setting, we have $\mathcal{Y} = \sigma(\mathcal{X}, \mathcal{Y})$-continuous linear functionals.

Apply this for example to the case that $\mathcal{Y}$ is a normed space, so $\mathcal{Y}^*$ is the Banach space of its norm-continuous functionals equipped with $\|\phi\|$, however, with $\sigma(\mathcal{Y}^*, \mathcal{Y}^{**})$-top, $\mathcal{Y}^*$ is a l.c. TVS.

By the above theorem, the w^*-continuous functionals on $\mathcal{Y}$ are precisely the the elements of $\mathcal{Y}$, viewed inside $\mathcal{Y}^{**}$.

Let X be compact Hausdorff (top) space.

Define $P(X) = \{\phi : C(X) \to \mathbb{C} \mid \phi \text{ bd, linear, } \|\phi\| \leq 1, \phi(1) = 1\}$.

Note: $P(X) = \{\phi \in C(X)^* \mid \|\phi\| \leq 1, \phi(1) = 1, \phi \text{ positive } (f \geq 0 \Rightarrow \phi(f) \geq 0)\}$.

Why? If ϕ linear and positive, we know from result in MAT4410 that ϕ is bd with $\|\phi\| = \phi(1)$, thus $\supseteq$.

On the other hand, if ϕ is in the first set for $P(X)$, let $f \geq 0$. Then $0 \leq \|f\| \cdot 1 - f \leq \|f\|$ and from $|\phi(\|f\| \cdot 1 - f)| \leq \|\phi\| (\|f\| \cdot 1 - f)$ it follows that $0 \leq f \leq \phi(f)$.

Let δ_x, $x \in X$ be the Dirac point mass $\delta_x : C(X) \to \mathbb{C}$, $\delta_x(f) = f(x)$ for $f \in C(X)$

<u>Proposition</u> $P(X) = \overline{\text{conv}}^{w^*} \{\delta_x \mid x \in X\}$.

<u>Proof</u>: For the inclusion $\supseteq$ it suffices to show that $P(X)$ is w^*-closed. For this, we show that

$$(C(X)^*, w^*\text{-top}) \xrightarrow{\Psi} \mathbb{C}, \quad \Psi(\Phi) = \Phi(1)$$

is continuous. The claim will follow since $P(X) = \Psi^{-1}(\{1\})$.

By description of w^*-continuity using nets, we must show that $\Psi(\Phi_\alpha) \to \Psi(\Phi)$ whenever $\Phi_\alpha \to \Phi$ in w^*-topology, for $\{\Phi_\alpha\}_{\alpha \in I}$ a net in $C(X)^*$. But w^*-convergence is pointwise convergence, so certainly $\Phi_\alpha(1) \to \Phi(1)$, as needed.

(because $\Phi_\alpha(f) \to \Phi(f)$ for every $f \in C(X)$).

To finish the proof, assume to reach a contradiction that there is $\Phi \in P(X) \setminus K$, where $K = \overline{\text{conv}}^{w^*} \{\delta_x \mid x \in X\}$.

Fact: it can be proved that there exist open sets U, V in $\mathcal{X} = C(X)^*$ s.t. $K \subseteq V$, $x \in U$, $U \cap V = \emptyset$.
(Proof: later).
By the Hahn-Banach separation theorem there exist $\tilde{\Phi} : C(X)^* \to \mathbb{C}$ w^*-continuous and $t \in \mathbb{R}$ s.t.

(1) $\operatorname{Re} \tilde{\Phi}(\psi) < t < \operatorname{Re} \tilde{\Phi}(\eta)$ for all $\psi \in K$ and $\eta \in U$.

Since $(C(X)^*, w^*)^* = C(X)$, may assume $\tilde{\Phi} = f \in C(X)$.

Thus, $\forall x \in X$, $\operatorname{Re} f(x) < t < \operatorname{Re} \Phi(f)$ (by (1)).

Replacing f with $\operatorname{Re} f$ we may assume f is $\mathbb{R}$-valued, and replacing f with $f + \|f\|$ we may assume $f \geq 0$.
In all, we get
$$\sup_{x \in X} f(x) < t < \Phi(f) \leq \|\Phi\| \cdot \|f\|, \text{ a contradiction}$$

A second application of the Hahn-Banach separation theorem

Proposition Let $\mathcal{X}$ be a normed space and $C \subseteq \mathcal{X}$ convex set. Then $\mathcal{X}$ is normed closed if and only if it is weakly closed.

Proof Assume C weakly closed. Let $\{c_\lambda\} \subseteq C$ with $\|c_\lambda - x\| \to 0$ for $x \in \mathcal{X}$. Then $c_\lambda \to x$ weakly, so $x \in C$ by assumption. Thus C normed closed.
Now show $\mathcal{X} \setminus C$ is weakly open. Let $x \in \mathcal{X} \setminus C$. By C open in norm $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset X \setminus C$, $B_\varepsilon(x) = \{y \mid \|y - x\| < \varepsilon\}$.
Hahn-Banach gives $\varphi \in \mathcal{X}^*$, $t \in \mathbb{R}$ s.t. $\operatorname{Re} \varphi(w) < t \leq \operatorname{Re} \varphi(z)$, $\forall z \in C, w \in B_\varepsilon(x)$. Since φ is weakly continuous, $\{\operatorname{Re} \varphi(w) < t\}$ is open, contains x and is contained in $\mathcal{X} \setminus C$.