Another applic. of convexity.

("Motivating Question 1")

X : cpt top. sp., $T : X \to X$ homeo.

$\Rightarrow \exists$ T-invariant (regular) Borel prob meas μ.

$$\int f(x)\, d\mu(x) = \int f(Tx)\, d\mu(x)$$

$$\overset{\shortparallel}{\phi_\mu(f)} \rightsquigarrow \overset{\shortparallel}{\phi_\mu(T^\# f)} \quad \text{for}$$

$$(T^\# f)(x) = f(Tx)$$

"Rough" idea was:

1. Take any μ_1 on X prob. meas!

   or $\phi_1 \in C(X)^*$, $\quad (*) \begin{cases} \phi_1(f) \geq 0 & (f \geq 0) \text{ "pos"} \\ \phi_1(1) = 1 \end{cases}$

   $\overset{\shortparallel}{\phi_{\mu_1}}$

2. form $\phi_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k_\# \phi_1$ "Cesàro mean" (satisfy $(*)$)

   with $(T_\# \phi)(f) = \phi(T^\# f)$ $\quad (T_\# \phi_\mu = \phi_{T_\# \mu})$

3. "w*-limit" $\phi_\infty = \lim \phi_n$ would be

   T-invariant, satisfies conditions $(*)$

Problem (about 3)

We don't know if $(\phi_n)_n$ is (w*) convergent

But: $K = \{\phi \in C(X)^*, \text{ pos}, (\Rightarrow \phi(1) = 1)\}$

is w*-cpt. $\Rightarrow$ $(\phi_n)_n$ has (many) accumulation points

$\rightsquigarrow$ enough to choose "$\overset{\text{conv.}}{\text{subseq}}$" $(\phi_{n_k})_{k=1}^\infty$ s.t.

$(T^\# \phi_{n_k})_k$ is also conv. (to same lim).

But: we need $\underline{\text{nets}}$ to understand w*-top.

(not seq.)

What we know:

$T_\# \phi_n - \phi_n = \frac{1}{n}(T^n{}_\# \phi_1 - \phi_1)$ has small norm

$(\leq \frac{2}{n})$

$\rightsquigarrow$ seqs $(\phi_n)_n$ & $(T_\# \phi_n)_n$ are asymptotically same.

$\forall f \in C(X)$ $\quad \phi_n(f) - T_\# \phi_n(f) \to 0 \quad (n \to \infty)$

$(O(\frac{1}{n}\|f\|))$

$\mathfrak{X} = \ell_\infty \mathbb{N} / c_0(\mathbb{N})$. quot. Ban. sp.

Note: $c_0(\mathbb{N})$ is a closed subsp. of $\ell_\infty \mathbb{N}$

for the norm $\|(a_n)_n\| = \sup |a_n|$

Take $\omega \in \mathfrak{X}^*$, $\quad \omega(1) = 1 \quad$ ($\because$ Hahn-Ban.)

i.e. func. on $\ell_\infty \mathbb{N}$ s.t. $\lim a_n = 0 \Rightarrow \omega((a_n)_n) = 0$

we can write $\omega((a_n)_n) = \lim\limits_{n \to \omega} a_n$

- const. seq. $\alpha = a_n \Rightarrow \lim\limits_{n \to \omega} a_n = \alpha$
- vanish on c_0-seq.
- linearity $\lim\limits_{n \to \omega} a_n + b_n = \lim\limits_{n \to \omega} a_n + \lim\limits_{n \to \omega} b_n$

Put $\phi_\omega(f) = \omega((\phi_n(f))_n)$. for $f \in C(X)$.

- $\|\phi_\omega\| = 1$, $\phi_\omega(1) = 1$ $\quad (\Rightarrow \phi_\omega(f) \geq 0$ for $f \geq 0)$

- $\phi_\omega(f) = \phi_\omega(T^\# f)$

diff: $\omega\Big(\underbrace{(\phi_n(f) - (T_\# \phi_n)(f))_n}_{c_0\text{-seq.}}\Big)$

$\rightsquigarrow$ corresp. Borel prob. meas. μ_ω is T-inv.