2019.02.18 no. 1

Summary

- Recap on Hilbert spaces
  - Convex sets, shortest element
- Operators on Hilbert spaces
  - Adjoint operator
  - Normal, self adjoint, positive ops.

Hilbert spaces

A <u>Hilbert space</u> is

- Banach sp. $\mathcal{H}$
- Hermitian inn. prod. $(u, v) \in \mathbb{C}$

  for $u, v \in \mathcal{H}$.

  - $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$
  - $(v, u) = \overline{(u, v)}$.
  - $(u, u) \geq 0$, equality iff $u = 0$
  - $\|u\| = \sqrt{(u, u)}$

Rem. • Parallelogram law $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

- Cauchy - Bunyakovsky - Schwarz ineq.

  $|(u, v)| \leq \|u\| \|v\|$

- $(u, v) = \sum_{k=0}^{3} \frac{\sqrt{-1}^k}{4} \|u + \sqrt{-1}^k v\|^2$
-
- W/o cpltness assumption pre Hilbert sp.

  $\xrightarrow[\text{completion}]{}$ Hilb. sp

Ex. • $l_2(\mathbb{N}) = \{(a_n)_{n=1}^{\infty} : \sum |a_n|^2 < \infty\}$

$u = (a_n)_n, \ v = (b_n)_n \rightsquigarrow (u, v) = \sum_n a_n \overline{b_n}$

• $L^2(\mathbb{R})$ "=" $\{f : \mathbb{R} \to \mathbb{C}$ meas. $\int |f(t)|^2 dt < \infty\}$

$(f, g) = \int f(t) \overline{g(t)} dt$

X

Thm. $\mathcal{H}$: Hilb sp $\rightsquigarrow$ $\exists$ orthonormal basis

$(u_i)_{i\in I}$ (I : some ind. set)

i.e. • $(u_i, u_j) = \delta_{ij}$

• span of $(u_i)_{i\in I}$ dense in $\mathcal{H}$.

$\rightsquigarrow$ $\forall$ $v \in \mathcal{H}$ can be written as $\sum \alpha_i u_i$

$\alpha_i = (v, u_i)$

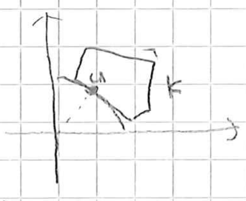
($\ell_2(I) \xrightarrow{\sim} \mathcal{H}$)

Riesz $\rightsquigarrow$ $\overline{\mathcal{H}}$

$\because$ Gram-Schmidt orthonormalization.

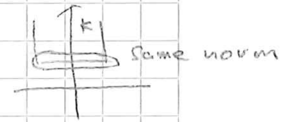
Thm $K \subset \mathcal{H}$ nonempty closed convex set.

$\exists!$ $u \in K$: $\|u\| = \min_{v\in K} \|v\|$.



cf. ℓ^2_∞ (Manhattan dist)

$\|(x,y)\| = \max(|x|, |y|)$



Proof : Step 1 Pick seq $(u_n)_n$ in K

s.t. $\|u_n\| \to \inf_{v\in K} \|v\|$

Step 2 $(u_n)_n$ is convergent

Para. law implies

$$\|u_m - u_n\|^2 = 2(\|u_m\|^2 + \|u_n\|^2) - 4\|\underbrace{(u_m+u_n)/2}_{\text{in } K}\|^2$$

$\simeq$ $m, n \gg 1$ $\quad$ bigger than

$4 \inf_{v\in K} \|v\|^2$ $\quad$ $4 \inf \|v\|^2$

very small when $m, n \gg 1$

$\Rightarrow$ $(u_n)_n$ is a Cauchy seq.

Step 3 $u = \lim_{n\to\infty} u_n$ satisfies $\|u\| = \min_{v\in K} \|v\|$

Step 4 If $u, u' \in K$ satisfy $\|u\| = \|u'\| = \min_{v\in K} \|v\|$

then $u = u'$

$$\|\underbrace{\frac{u+u'}{2}}_{\text{in } K}\|^2 = \frac{1}{2}(\|u\|^2 + \mathrm{Re}(u,u')) \overset{(*)}{\leq} \|u\|^2 = \min_v \|v\|^2$$

$\Rightarrow$ must have eq. at $(*)$ $\Rightarrow \mathrm{Re}(u,u') = \|u\|^2$ $\Rightarrow (u,u') = \|u\|^2$

2019.02.18 no 3

Operators on Hilb. sp.

$\mathcal{L}(\mathcal{H}) = (B(\mathcal{H})) \ni \{T : \mathcal{H} \to \mathcal{H} \text{ bdd}\}$

adjoint op T^* of $T \in \mathcal{L}(\mathcal{H})$:

characterized by $(Tu, v) = (u, T^*v)$

i.e. T^*v reps (bdd) func $u \mapsto (Tu, v)$

Rem $(u_i)_i$ onb of $\mathcal{H}$ $\Rightarrow$ $X_{ij} = (Tu_j, u_i)$

rep. T : $v = \sum \alpha_j u_j \Rightarrow Tv = \sum_{i,j} X_{ij} \alpha_j u_i$

T^* is rep'ed by $(X^*)_{ij} = \overline{X_{ji}}$

T is normal if $TT^* = T^*T$

self adj if $T = T^*$ skew ⋯ $T^* = -T$

positive if $(Tu, u) \geq 0$ $\forall u \in \mathcal{H}$

Prop. Pos. $\Rightarrow$ self adj.

$\because$ Enough to prove $(Tu, v) = (u, Tv)$

$$(Tu, v) = \frac{1}{4} \sum_{k=0}^{3} \sqrt{-1}^{k} (T(u + \sqrt{-1}^{k} v), u + \sqrt{-1}^{k} v)$$

$$(u, Tv) = \frac{1}{4} \sum \sqrt{-1}^{k} (u + \sqrt{-1}^{k} v, T(u + \sqrt{-1}^{k} v)).$$

Rem. $(\alpha T)^* = \overline{\alpha} T^*$, $(T+S)^* = T^* + S^*$.

$T^{**} = T$

Ex. Diag. op. on $\ell_2(\mathbb{N})$

$(\alpha_n)_{n=1}^{\infty} \in \ell_\infty(\mathbb{N})$ $(\sup_n |\alpha_n| < \infty)$

$T : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$, $(b_n)_n \mapsto (\alpha_n b_n)_n$.

T^* is from $(\overline{\alpha_n})_n$.

Ex. $\mathcal{H} = L^2([0, 1])$.

$(Tf)(t) = t f(t)$.

Ex. T^*T is positive.

Outlook of spectral theory :

1. Normal op. generates function alg on $\sigma(T)$ "spectrum of T" $\subset \mathbb{C}$.

2. T self adj $\Longleftrightarrow$ $\sigma(T) \subset \mathbb{R}$

3. positivity for ops $\leftrightarrow$ pos. defness of inn. prod.

(we'll turn op. alg into pre Hilb. sp.)

Rem We'll also consider <u>unbounded</u> self adj ops (but need to be careful)

$L^2([0,1])$, $L^2(\mathbb{R})$ have

$T = \sqrt{-1}\,\frac{d}{dt}$; def'ed for f s.t. $f' \in L^2$.

$(Sf)(t) = t f(t)$ is ubdd on $L^2(\mathbb{R})$

$\Delta = -\frac{d^2}{dt^2}$ on $L^2(\mathbb{R})$, $-\sum_{i=1}^{n} \frac{d^2}{dx_i^2}$ on $L^2(\mathbb{R}^n)$

pos. (unbdd).

Conjugate sp.

$\overline{\mathcal{H}} = \{\bar{u} : u \in \mathcal{H}\}$ • $\overline{\alpha u} = \bar{\alpha} . \bar{u}$.

• $(\bar{u}, \bar{v}) = (v, u)$ $\left(= \overline{(u, v)}\right)$

Riesz rep. th'm.

$\forall \phi : \mathcal{H} \to \mathbb{C}$ bdd $\exists\, v \in \mathcal{H}$ s.t. $\phi(u) = (u, v)$

$\phi_v(u) = (u, v)$: $\overline{\mathcal{H}} \longrightarrow \mathcal{H}^*$, $v \mapsto \phi_v$ is iso.