Summary

- Unitary ops
- Projections
  - closed subspaces
- Kernel, range & adjoint
- Isometry / coisometry

Unitary operator. $U \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$: $U^* = U^{-1}$

Ex. bilateral shift $\mathcal{H} = \ell_2 \mathbb{Z} = \{(a_n)_{n=-\infty}^{\infty} \mid \Sigma |a_n|^2 < \infty\}$

$v = (a_n)_n$ $Uv = (b_n)_n$ $b_n = a_{n-1}$

(or $U\delta_n = \delta_{n+1}$)

$U^* v = (c_n)_n$ $c_n = a_{n+1}$ $U^*\delta_n = \delta_{n-1}$

(Ex. check this)

Unitary equivalence relation.

- $S \in \mathcal{L}(\mathcal{H})$, $T \in \mathcal{L}(\mathcal{H}')$ are unitarily equiv. if $\exists U \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ unitary $T = USU^*$
- X set, $\pi: X \to \mathcal{L}(\mathcal{H})$, $\pi': X \to \mathcal{L}(\mathcal{H}')$ map. (usually hom from an alg) are uni. equiv. if $\exists U$ unitary $\pi'(x) = U\pi(x)U^*$.

Ex. $\dim \mathcal{H} = n < \infty$. normal $T \in \mathcal{L}(\mathcal{H})$ up to uni. eq.: classified by $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ eigenvals counted w/ multiplicit

Projection $P \in \mathcal{L}(\mathcal{H})$: $P^2 = P = P^*$.

$M^\perp$)

Prop. $\{$prjs in $\mathcal{L}(\mathcal{H})\} \overset{1:1}{\longleftrightarrow} \{$closed subsps. $M \subset \mathcal{H}\}$

$P \mapsto \operatorname{Ran} P$.

P_M: orthogonal prj. to $M \leftarrow M$

$\because$ We'll check $P_M^2 = P_M = P_M^*$

$M \subset \mathcal{H}$ subsp. $\rightsquigarrow$ complement $M^\perp = \{v \in \mathcal{H} :$
$\forall u \in M \quad (u, v) = 0\}$

- $(M^\perp)^\perp$ = closure of M (so M clos $\Rightarrow M = M^{\perp\perp}$)
- M clos. $\Rightarrow \mathcal{H} = M \oplus M^\perp$ i.e.
$\forall u \in \mathcal{H} \ \exists!\ v \in M,\ v' \in M^\perp \quad u = v + v'$

$P_M : \mathcal{H} \to M, \quad v + v' \mapsto v$

Step 1. $P_M^2 = P_M$: $P_M(v) = v$ if $v \in M$

Step 2 $P_M^* = P_M$ $\quad u_i = v_i + v_i' \quad (i = 1, 2)$

$(P_M u_1, u_2) = (v_1, u_2) \underset{v_2' \in M^\perp}{=} (v_1, v_2)$

$(u_1, P_M u_2)$ similar

Step 3 $(\mathrm{Ran}\, P)^\perp = \mathrm{Ker}\, P$.

$(Pu, v') = 0 \underset{P = P^*}{\Leftrightarrow} (u, Pv') = 0$

Step 4 $P_{\mathrm{Ran} P} = P$

$\mathcal{H} = \mathrm{Ran}\, P \oplus (\mathrm{Ran}\, P)^\perp \underset{\mathrm{St.}\, 3}{=} \mathrm{Ran}\, P \oplus \mathrm{Ker}\, P$

P_1, P_2 are <u>orthogonal</u> : $P_1 P_2 = 0$ $(= P_2 P_1)$

$(P_i)_{i \in I}$ are <u>mutually</u> <u>orth</u>. : $\forall i \neq j \quad P_i P_j = 0$

Prop. $(P_i)_{i=1}^n$ prjs, $M_i = \mathrm{Ran}\, P_i$

$\mathrm{Id} = P_1 + \dots + P_n \Leftrightarrow \mathcal{H} = \underbrace{M_1 \oplus \dots \oplus M_n}_{\text{orth. dir. sum}}$

Proof $\Leftarrow$: Assumption : $\forall u \in \mathcal{H} \ \exists!\ v_i \in M_i$
$u = v_1 + \dots + v_n \qquad M_i \perp M_j \quad (i \neq j)$

$P_i u = v_i \rightsquigarrow (\Sigma P_i) u = \Sigma v_i = u$

$\Rightarrow$: By ind. we may assume $n = 2$, and
prove $P_1 + P_2$ prj $\Rightarrow \mathrm{Ran}(P_1 + P_2) = \mathrm{Ran}\, P_1 \oplus \mathrm{Ran}\, P_2$.

Step 1 $u \in \mathrm{Ran}\, P_1 \Rightarrow P_2 u = 0$ $(\mathrm{Ran}\, P_1 \perp \mathrm{Ran}\, P_2)$

$\because ((P_1+P_2)u, u) = ((P_1+P_2)^2 u, u) = ((P_1+P_2)u, \sim)$

$= \sum_{i,j=1}^{2} (P_i u, P_i u) = (P_1 u, u) + 3(P_2 u, u) = ((P_1+P_2)u, u) + 2\|P_2 u\|^2.$

$P_1 u = u$

$P_2 = P_2^*$

$\Rightarrow \|P_2 u\| = 0$

Step 2 $\mathrm{Ran}\, P_1 \oplus \mathrm{Ran}\, P_2 = \mathrm{Ran}\,(P_1+P_2)$

$\because$ Step 1 impls $P_1 P_2 = 0 = P_2 P_1$

$\Rightarrow P_1 u + P_2 v = (P_1 + P_2)(P_1 u + P_2 v)$

$\Rightarrow \mathrm{Ran}\, P_1, \mathrm{Ran}\, P_2 \subset \mathrm{Ran}\,(P_1+P_2)$

Ker, Ran, adjoint (for non-normal ops)

Prop. $T \in \mathcal{L}(\mathcal{H})$: $\mathrm{Ker}\, T = (\mathrm{Ran}\, T^*)^{\perp}$

Proof $\subset$: $u \in \mathrm{Ker}\, T \Rightarrow \forall v$

$0 = (Tu, v) = (u, T^* v)$

$\Rightarrow u \perp \mathrm{Ran}\, T^*$

$\supset$: $u \in (\mathrm{Ran}\, T^*)^{\perp}$ means $\forall v$ $(Tu, v) = 0$

with $v = Tu$ $\|Tu\| = 0$

Isometry / coisometry

$V \in \mathcal{L}(\mathcal{H})$ is <u>isometry</u> if $\|Vu\| = \|u\|$ $\forall u \in \mathcal{H}$.

Prop. V isometry $\Leftrightarrow V^* V = \mathrm{Id}$

Proof $\Leftarrow$: $(V^*V u, u) = \|Vu\|^2$

$\Rightarrow (V^*V u, u') = \frac{1}{4} \sum_{k=0}^{3} \sqrt{-1}^{k} \underbrace{(V^*V(u+\sqrt{-1}^k u'), u+\sqrt{-1}^k u')}_{= (V(u+\sqrt{-1}^k u'), V(u+\sqrt{-1}^k u'))}$

$= \frac{1}{4} \sum \sqrt{-1}^k \|u + \sqrt{-1}^k u'\|^2 = (u, u')$

$\Rightarrow V^* V u = u$

V is a coisometry if V^* is isomet.

$(\Leftrightarrow \; VV^* = I_{\mathcal{H}})$

V is a partial isometry if $\|Vu\| = \|u\|$ for $u \in (\mathrm{Ker}\, V)^\perp$.

$(\Leftrightarrow \; V^*V, \; VV^*$ are prjs $)$.

Ex. unilateral shift. $\mathcal{H} = \ell_2 \mathbb{N}$.

$u = (a_n)_{n=1}^\infty \Rightarrow Vu = (b_n)_n \quad b_n = a_{n-1}, \; b_1 = 0$

$($i.e. $\delta_n \mapsto \delta_{n+1})$

$V^* : (a_n)_n \mapsto (c_n)_n \; ; \; c_n = a_{n+1}$

$\mathrm{Ker}\, V^* = \mathbb{C} \cdot \underset{\delta_1}{(1, 0, 0, \ldots)} = (\mathrm{Ran}\, V)^\perp$.

Ex. $\mathcal{H} = H^2 \overset{\text{Hardy space}}{=} \{ f : \overline{\mathbb{D}} \to \mathbb{C}$, holom on $\mathbb{D}$, L^2 on $\mathbb{T} = \partial \overline{\mathbb{D}} \}$

$\overline{\mathbb{D}} = \{ z \in \mathbb{C} : |z| \leq 1 \}$

$(T_z f)(z) = z f(z)$ isometry on H^2.

V & T_z unitary equivalent.

$U : \ell_2 \mathbb{N} \to H^2, \; (a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n z^{n-1}$