Summary

- Spectrum of ops
  - self adj ops have real sp.
  - pos. ops have nonneg. sp.

Notation

$\mathcal{H}$: Hilb. sp. , $T, S, \cdots \in \mathcal{L}(\mathcal{H})$

The <u>resolvent</u> <u>set</u> of T is

$$\rho(T) = \{\lambda \in \mathbb{C} \ ; \ T - \lambda \, \mathrm{Id}_{\mathcal{H}} \text{ is invertible}\}$$

The <u>Spectrum</u> of T is

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

Ex. • $\mathcal{H} = \mathbb{C}^n$ fin. dim case.

$\sigma(T)$ = eigenvals of T.

$\because$ $T - \lambda I_n$ inv $\Leftrightarrow$ $\det(T - \lambda I_n) \neq 0$

$\Leftrightarrow$ λ not eigenval. of T

• Proj. P has $\sigma(P) \subset \{0, 1\}$ non.eq. iff $P = 0, \mathrm{Id}$.

$\because$ $\mathcal{H} = \mathrm{Im}\, P \oplus \mathrm{Ker}\, P$ orth. dec.

$P - \lambda \, \mathrm{Id}_{\mathcal{H}}$ acts as

$(1 - \lambda)\, \mathrm{Id}_{\mathrm{Im}\, P}$ on $\mathrm{Im}\, P$

$-\lambda \ \mathrm{Id}_{\mathrm{Ker}\, P}$ on $\mathrm{Ker}\, P$.

$\lambda \neq 0, 1$ $\Rightarrow$ Both are inv'ble.

<u>Operator</u> <u>norm</u> of T :

$$\|T\| = \sup\{\|Tu\| \ ; \ u \in \mathcal{H}, \ \|u\| \leq 1\}$$

satisfies $\|ST\| \leq \|S\| \cdot \|T\|$, $\|\mathrm{Id}\| = 1$.

Prop. $\|T\| < 1 \Rightarrow \mathrm{Id} - T$ is inv'ble

Proof. Step 1 $\|T^k\| \le \|T\|^k$

Step 2 $\sum_{k=0}^{\infty} T^k$ is convergent $(T^0 = Id)$

Step 3 $\sum_{k=0}^{\infty} T^k = (Id - T)^{-1}$

$\because$ $(Id - T)\sum_{k=0}^{N} T^k = Id - T^{N+1} \to Id \quad (N\to\infty)$

Cor. $GL(\mathcal{H}) = \{ S \in \mathcal{L}(\mathcal{H}) \text{ inv'le} \}$ is norm open.

$\because$ $S \in GL(\mathcal{H})$, $\|S' - S\| < \frac{1}{\|S^{-1}\|} \Rightarrow S' \in GL(\mathcal{H})$

from $\|S'S^{-1} - Id\| < 1$. $(\Rightarrow S'S^{-1}$ inv.le)

Cor of Cor. $\rho(T)$ is open ($\Leftrightarrow \sigma(T)$ is closed)

Cor (of Prop). $\lambda \in \sigma(T) \Rightarrow |\lambda| \le \|T\|$

$\because$ Will show $\lambda > \|T\| \Rightarrow \lambda \in \rho(T)$.

$\lambda > \|T\| \Rightarrow \|\lambda^{-1} T\| < 1 \Rightarrow Id - \lambda^{-1}T$ inv'le

$\Rightarrow T - \lambda Id = -\lambda(Id - \lambda^{-1}T)$ inv'le

Another invertibility criterion.

Prop 2 T is inv'le iff

1) T is bdd below ; $\exists\, \varepsilon > 0 : \|Tu\| \ge \varepsilon \|u\|$.

2) T has dense range.

Proof "$\Rightarrow$" 1) $\varepsilon = \|T^{-1}\|^{-1}$ from $u = T^{-1}v$

$\|T^{-1}\|\|v\| \ge \|T^{-1}v\|$. , 2) $\operatorname{Im} T = \mathcal{H}$.

"$\Leftarrow$" Step 2 $\operatorname{Im} T = \mathcal{H}$

2) impls $\forall u\ \exists v_n \in \mathcal{H}$ s.t. $Tv_n \to u$.

$\|v_m - v_n\| \le \varepsilon^{-1}\|Tv_m - Tv_n\| \to 0 \quad (m, n\to\infty)$.

So $(v_m)_m$ is Cauchy. $u = T(\lim v_m)$

Step 1 T is inj.

Step 3 T^{-1} (exists as lin. map by St. 1 & 2)
is bdd.
$\|T^{-1}v\| \leq \varepsilon^{-1}\|v\|$ (take $u = T^{-1}v$).

Thm. 1) $S = S^* \Rightarrow \sigma(S) \subset \mathbb{R}$ $([-\|S\|, \|S\|])$

2) T pos. $\sigma(T) \subset [0, \|T\|]$

Proof. 1)

Step. 1 Enough to show $\sqrt{-1}t \in \rho(S)$ for $t \neq 0$ real.

$\because S - (s + \sqrt{-1}t)\mathrm{Id} = \underbrace{(S - s\mathrm{Id})}_{\text{self adj.}} - \sqrt{-1}t\mathrm{Id}$

for $s, t \in \mathbb{R}$.

Step 2. $\|(S - \sqrt{-1}t\mathrm{Id})u\| \geq |t|\,\|u\|$.

$\because$ $((S - \sqrt{-1}t)u, u) = \underbrace{(Su, u)}_{\text{real}} - \sqrt{-1}t(u, u)$

- abs. val $\geq$ |img. part| $= |t|\,\|u\|^2$ --- (#)
- $\|(S - \sqrt{-1})u\| \cdot \|u\| \underset{\text{CBS}}{\geq}$ abs. val

Step 3. $S - \sqrt{-1}t\mathrm{Id}$ has dense img.

$\because$ Enough to see $(\mathrm{Im}(S - \sqrt{-1}t\mathrm{Id}))^{\perp} = 0$.

Take $u \in (\mathrm{Im}(\cdots))^{\perp}$.

Then $((S - \sqrt{-1}t\mathrm{Id})u, u) = 0 \underset{\text{above (\#)}}{\Longrightarrow} \|u\| = 0$.

2). We already know $\sigma(T) \subset [-\|T\|, \|T\|]$.

Want : $a < 0 \Rightarrow T - a\mathrm{Id}$ inv.le.
$(= T + |a|\mathrm{Id})$

$((T - a\mathrm{Id})u, u) = \underbrace{(Tu, u)}_{\text{pos.}} - a(u, u) \geq |a| \cdot \|u\|^2$

We can do analogue of St 2 & 3. $\square$