Summary

- Banach algebras
  - Multiplicative functionals
  - Gelfand transform.

Banach algebra $\mathcal{A}$ is

- Banach sp. $\leadsto$ $\|x\|$ makes sense for $x \in \mathcal{A}$. ...
- $\mathbb{C}$-algebra : product ab makes sense
  $(ab)c = a(bc)$ for $a, b, c \in \mathcal{A}$
  $(\lambda a)b = a(\lambda b)$, $(a+a')b = ab + a'b$. ...
  $\lambda \in \mathbb{C}$.
- submultiplicativity $\|ab\| \leq \|a\| \cdot \|b\|$

Examples 1 $C(X)$ X : cpt top. sp.

$\|f\| = \max_{x \in X} |f(x)|$. $\|fg\| \leq \|f\| \cdot \|g\|$

2 $\mathcal{L}(\mathcal{H})$ for Hilb sp. $\mathcal{H}$.

$\leadsto$ closed subalgebras $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ also

$\mathcal{K}(\mathcal{H}) = \{$cpt. ops$\}$, ...

$T \in \mathcal{L}(\mathcal{H})$ $\leadsto$ closure of $\{P(T) : P(x)$ polynom$\}$

Convention : we'll assume $\mathcal{A}$ is unital

(has mult. unit $1_{\mathcal{A}}$ or $\mathbb{C} \subset \mathcal{A}$.)

So exclude $c_0(\mathbb{N})$, $\mathcal{K}(\mathcal{H})$, ...

$\varphi \in \mathcal{A}^*$ is multiplicative (character) if

$\varphi(ab) = \varphi(a)\varphi(b)$, $\varphi(1) = 1$.

$M_{\mathcal{A}} = \{\varphi \in \mathcal{A}^* : \text{multiplicative}\}$

Ex. $\mathcal{A} = C(X)$. each $x \in X$ determines

$ev_x : \mathcal{A} \to \mathbb{C}$, $f \mapsto f(x)$

Thm. $\Phi : X \to M_{C(X)}$, $x \mapsto ev_x$ is homeo (for weak* - top. on $M_{C(X)}$)

Proof. Step 1 Continuity of Φ

$O_{f_1, \dots, f_n, A_1, \dots, A_n} = \{ \varphi : \varphi(f_i) \in A_i \}$

$f_i \in C(X)$, $A_i \subset \mathbb{C}$ open.

formed a base of open sets

inv. img by Φ : $\{ x \in X : f_i(x) \in A_i \}$

$= \bigcap \underbrace{f_i^{-1}(A_i)}_{\text{open in } X}$: open

Step 2. injectivity

$C(X)$ separates pts of X

Step 3 surjectivity

take $\varphi \in M_{C(X)}$, put $\mathcal{K} = \operatorname{Ker} \varphi \subset C(X)$

3-1 : $\exists\, x \in X$ s.t. $ev_x(\mathcal{K}) = 0$.

Otherwise : pick $f^x \in \mathcal{K}$ s.t. $f^x(x) \neq 0$

let U^x neigh. of x s.t. $f^x|_{U^x} > 0$

$(U^x)_{x \in X}$ open cover of X.

X cpt $\leadsto$ $\exists\, x_1, \dots, x_n$ s.t. $X = \bigcup_{k=1}^{n} U^{x_k}$

$\leadsto$ $g = \sum_{k=1}^{n} |f^{x_i}|^2$ is in $\mathcal{K}$.

pos. at $\forall x \in X$

$\Rightarrow$ $\min_{x \in X} g(x) > 0$ so g is inv'le.

$\underbrace{\varphi(g)}_{0} \varphi(g^{-1}) = 1$. contradiction

3-2 $\varphi = ev_x$ for x in 3-1.

$\because$ $\forall f$ $f - \varphi(f).1 \in \mathcal{K}$.

$f(x_0) - \varphi(f) = (f - \varphi(f) 1)(x_0) = 0.$

Step 4. Φ is homeo.

$\because$ we already know $\Phi : X \to M_{C(X)}$ is cont. bij. from cpt sp. to Hausdorff sp. $\square$

- - - - - - -

Motivation: we want to do something similar for commutative ($ab = ba$) Banach algs

$$\mathcal{A} \rightsquigarrow \underset{\substack{\text{cpt. top.}\\ \text{sp. (see next)}}}{M_{\mathcal{A}}} \rightsquigarrow \underset{\text{comm. Ban. alg again}}{C(M_{\mathcal{A}})}$$

Prop. $\mathcal{A}$ (unital) Banach alg.

$M_{\mathcal{A}}$ is w^*-cpt subset of $(\mathcal{A}^*)_1$.

Proof. Step 1 $M_{\mathcal{A}} \subset (\mathcal{A}^*)_1$

Take $\varphi \in M_{\mathcal{A}}$, put $\mathcal{K} = \operatorname{Ker} \varphi$.

$\forall a \in \mathcal{A}$ $\exists \lambda \in \mathbb{C}$, $b \in \mathcal{K}$ $a = \lambda + b$.

$(\lambda = \varphi(a),\ b = a - \varphi(a) \cdot 1.)$

$$\|\varphi\| = \sup_{a \neq 0} \frac{|\varphi(a)|}{\|a\|} = \sup_{\substack{b \in \mathcal{K} \\ \lambda + b \neq 0}} \frac{|\lambda|}{\|\lambda + b\|} \underset{\text{div. by } \lambda}{=} \sup_{b \in \mathcal{K}} \frac{1}{\|1 + b\|}$$

Claim $b \in \mathcal{K} \Rightarrow \|1 + b\| \geq 1$. $(\Rightarrow \|\varphi\| = 1)$

$\because$ $\|1 + b\| < 1 \Rightarrow b = -(1 - \underbrace{(1 + b)}_{\text{put } c})$ is inv.

$(1 - c)^{-1} = \sum_{k=0}^{\infty} c^k$ makes sense

Step 2 M_φ is w^*-closed in $(\mathcal{A}^*)_1$ $(\Rightarrow w^*\text{-cpt})$

$$M_\varphi = \underbrace{\bigcap_{a, b \in \mathcal{A}} \{\varphi(ab) = \varphi(a)\varphi(b)\}}_{w^*\text{-closed}} \cap \{\varphi(1) = 1\}$$

$\square$

$\mathcal{A}$ (comm) Banach alg.

The <u>Gelfand transform</u> of $\mathcal{A}$ is the map

$\Gamma : \mathcal{A} \to C(M_{\mathcal{A}})$, $a \mapsto (\varphi \mapsto \varphi(a))$.

Example. $\mathcal{A} = \ell_1(\mathbb{Z}) = \{ (a_n)_{n=-\infty}^{\infty} : \underbrace{\sum |a_n|}_{\| (a_n)_n \|_1} < \infty \}$

product of $a = (a_n)_n$ & $b = (b_n)_n$.

$(a * b)_n = \sum_{m+\ell = n} a_m b_\ell$. (convolution)

(interpret $(a_n)_n$ as $f^a(z) = \sum_{n=-\infty}^{\infty} a_n z^n$

$f^{a*b}(z) = f^a(z) f^b(z)$)

$\| a * b \|_1 = \sum_{n=-\infty}^{\infty} \left| \sum_{m=-\infty}^{\infty} a_m \underbrace{b_{n-m}}_{\ell} \right|$

$\leq \sum_{m, \ell \in \mathbb{Z}} |a_m| \cdot |b_\ell| = \|a\|_1 \cdot \|b\|_1$

$\rightsquigarrow \ell_1(\mathbb{Z})$ is a Banach alg.

unit. $\delta_0 = (\cdots 0 \underset{n=0}{1} 0 \cdots)$ $\quad f^{\delta_0}(z) = 1$.

$\|\delta_n\|_1 = 1$ for all n, $\delta_{-n} = \delta_n^{-1}$

$\varphi \in M_{\ell_1(\mathbb{Z})} \Rightarrow z_\varphi = \varphi(\delta_1)$ is invertible

$(z_\varphi^{-1} = \varphi(\delta_{-1}))$

$\|\varphi\| = 1 \Rightarrow |z_\varphi|, |z_\varphi^{-1}| \leq 1 \Rightarrow |z_\varphi| = 1$.

Any $z \in \mathbb{C}$, $|z| = 1$ will define cont. map $\varphi^z : \ell_1(\mathbb{Z}) \to \mathbb{C}$, $(a_n)_n \mapsto \sum a_n z^n$.

$\rightsquigarrow M_{\ell_1(\mathbb{Z})} \simeq \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$

Up to this, $\Gamma : \ell_1(\mathbb{Z}) \to C(\mathbb{T})$, $a \mapsto f^a$