Summary

Abstract spectrum

- spectrum is nonempty (over $\mathbb{C}$)
- (spectral radius)
- holom. func. calc.

$\mathcal{A}$: unital Banach alg. $\swarrow$ cpt, bdd by $\|a\|$ cf. 02.25

<u>Spectrum</u> of $a \in \mathcal{A}$ is $\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : a - \lambda$ not inv'le $\}$

(res. set : $\rho_{\mathcal{A}}(a) = \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$)

Rem. if $\mathcal{B}$ is a subalg of $\mathcal{A}$, $b \in \mathcal{B}$

$\rho_{\mathcal{B}}(b) \subset \rho_{\mathcal{A}}(a)$ ($\Leftrightarrow \sigma_{\mathcal{A}}(b) \subset \sigma_{\mathcal{B}}(b)$)

but not eq. in general.

Ex. $\mathcal{A} = C(\mathbb{T})$, $\mathcal{B}$: closure of $\{ \sum_{n=0}^{\infty} a_n z^n \}$.

$\sigma_{\mathcal{B}}(z) = \overline{\mathbb{D}}$ $\quad \sigma_{\mathcal{A}}(z) = \mathbb{T}$

Liouville's thm for Banach spaces

$\mathcal{X}$: Banach sp. $f : \mathbb{C} \to \mathcal{X}$ <u>holomorphic</u>

$\forall z \ \exists f_z \in \mathcal{X}$ s.t. $\dfrac{\| f(z+w) - w f_z \|}{|w|} \xrightarrow[w \to 0]{} 0$

$f'(z) := f_z$

"L." If f is bounded $\sup_z \| f(z) \| < \infty$

then f is const.

Proof 1 Step 1: $\varphi(f(z))$ is const. for any $\varphi \in \mathcal{X}^*$.

$\because$ usual hol. func. $\mathbb{C} \to \mathbb{C}$

Step 2 $f(z)$ is const.

$\because$ otherwise $f(z_1) - f(z_0) \neq 0$

$\Rightarrow$ HB impls $\exists \varphi \quad \varphi(f(z_1) - f(z_0)) \neq 0$

Proof 2 : use the prop. of <u>harmonic</u> funcs : "avg of f around z = $f(z)$"

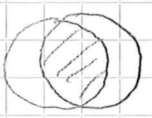
Step 1 $\forall z$ the func $0<r \mapsto \frac{1}{2\pi r^2}\iint_{|x+iy-z|<r} f(x+iy)\,dx\,dy$

is const.

$\because$ $\bar\partial_z f = 0 \Rightarrow \Delta^2 f = 0 \Rightarrow$ claim.

Step 2 call the above q'ty $(I(z) =) I(z,r)$

$$\| I(z_1, r) - I(z_2, r) \| = O\left(\frac{|z_1 - z_2|}{r}\right)$$

$\because$  big overlap.

Step 3 $I(z_1) = I(z_2)$ $\because$ take $r \to \infty$

Thm. $\mathcal{A}$: unital Banach alg over $\mathbb{C}$ $(0 \neq 1_{\mathcal{A}})$

$\forall a \in \mathcal{A}$ $\sigma_{\mathcal{A}}(a)$ is nonempty.

Proof Pf by contradiction

Put $f(z) = (a-z)^{-1}$.

Step 1. $f(z)$ is holomorphic.

$\because$ $f'(z) = (a-z)^{-2}$. from

$$(a-(z+w))^{-1} = \sum_{k=1}^{\infty} (a-z)^{-k} w^{k-1}$$

conv. for $\|(a-z)^{-1} w\| < 1$.

Step 2. $f(z)$ is bounded

$\because$ Enough to check $\lim_{z\to\infty} f(z) \to 0$

$f(z) = \frac{1}{z} \cdot (\frac{1}{z} a - 1)^{-1}$ & $\frac{1}{z} a \to 0$ $(z \to \infty)$

$\Rightarrow (\frac{1}{z} a - 1)^{-1} \to 1$.

Step 3 Liouville $\Rightarrow$ contra.

$f(0) = \lim f(z) = 0$ but $f(0) = a^{-1}$.

Cor (Gelfand - Mazur) $\mathcal{A}$ as above

If $\mathcal{A}$ is a skewfield $(\forall a \neq 0$ inv'le$)$

then $\mathcal{A} \simeq \mathbb{C}$

$\because$ pick any $\lambda \in \sigma(a)$ $a - \lambda$ not inv $\Rightarrow a - \lambda = 0$

Rem. Above thm / cor don't hold over $\mathbb{R}$

e.g. $\mathbb{C}$ as $\mathbb{R}$-Banach alg.

$\sigma'(\sqrt{-1}) = \{ t \in \mathbb{R} : \sqrt{-1} - t$ is not inv. $\}$

is empty. (and $\mathbb{C}$ is a field, $\not\cong \mathbb{R}$)

Maximal ideal $\longleftrightarrow$ mult. func.

Prop. $\mathcal{M} \subset \mathcal{A}$ nontriv. max. ideal $\Rightarrow$ closed

$\because$ open neigh. of 1 don't intersect w/ $\mathcal{M}$.

(inv'le elems)

Cor. $\mathcal{A}$ comm., $\mathcal{M} \subset \mathcal{A}$ nontriv. max ideal

$\Rightarrow \mathcal{A}/\mathcal{M} \simeq \mathbb{C}$.

$\because$ Enough to check $\mathcal{A}/\mathcal{M}$ is a skew field.

Comm alg : $0 \neq a$ noninv'le

$\Rightarrow (a) = \{ ba : b \in \mathcal{A} \}$ is

nontriv. ideal $(\not\ni 1)$

- $\varphi \in M_{\mathcal{A}} \Rightarrow \operatorname{Ker} \varphi$ max ideal.
- $\mathcal{M} \subset \mathcal{A}$ max id. $\Rightarrow \varphi_{\mathcal{M}} : \mathcal{A} \to \mathcal{A}/\mathcal{M} \xrightarrow{\sim} \mathbb{C}$ is multiplicative.
- Zorn $\exists$ max id.

Spectral radius of $a \in \mathcal{A}$.

$$r_{\mathcal{A}}(a) = \max_{z \in \sigma_{\mathcal{A}}(a)} |z|$$

later.

Prop. $\mathcal{A}$ unital comm. $M = M_{\mathcal{A}}$

$a \in \mathcal{A}$ is inv. iff $\Gamma(a) \in C(M)$ is so.

$(\rightsquigarrow \sigma_{\mathcal{A}}(a) = \sigma_{C(M)}(\Gamma(a)))$

Pf. a inv $\Rightarrow \Gamma(a)$ inv easy $(\Gamma(a^{-1}) = \Gamma(a)^{-1})$

a not inv. $\Rightarrow \Gamma(a)$ not inv.

$\because$ a not inv. $\Rightarrow (a) \neq \mathcal{A} \Rightarrow \exists$ max $\mathcal{M}$

containing (a) (Zorn) $\Rightarrow \varphi_{\mathcal{M}}(a) = 0$

Holom. func. calc. : $f : \mathbb{C} \to \mathbb{C}$ holom.

$\mapsto f(z) = \sum_{n=0}^{\infty} a_n z^n$ rad. conv. $= \infty$

$\left(\overline{\lim} \sqrt[n]{|c_n|} = 0 \right)$

$a \in \mathcal{A} \Rightarrow f(a) = \sum_{n=0}^{\infty} c_n a^n$ is convergent.

$\because \| c_n a^n \| = o(\varepsilon^n)$ for any $\varepsilon > 0$

$\mathcal{A}$ unital Ban. alg.

Prop. $\sigma(f(a)) = f(\sigma_{\mathcal{A}}(a))$

Step 1 Repl $\mathcal{A}$ by comm $\mathcal{B}$ s.t.

$\sigma_{\mathcal{B}}(a) = \sigma_{\mathcal{A}}(a)$

$\because \mathcal{B} =$ clos. of $(1, a, (a-z)^{-1}\ z \in \rho_{\mathcal{A}}(a))$

Step 2 $\sigma(f(a)) = f(\sigma(a))$

$\because \sigma(f(a)) = \Gamma(f(a))(M)$ by Prop.

$\Gamma(f(a))(M) = f(\Gamma(a))(M)$

$f(\Gamma(a))(M) = f(\Gamma(a)(M))$ from formula.

$\Gamma(a)(M) = \sigma(a)$ again by Prop.

Th'm $r_{\mathcal{A}}(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}}$

Step 1 WMA $\mathcal{A}$ comm.

Step 2 $r_{\mathcal{A}}(a) \leq \underline{\lim} \|a^n\|^{\frac{1}{n}}$.

$\because r_{\mathcal{A}}(a)^n = r_{\mathcal{A}}(a^n) \leq \|a^n\|$.

Step 3 $\overline{\lim} \|a^n\|^{\frac{1}{n}} \leq r_{\mathcal{A}}(a)$

$\because G(z) = (a-z)^{-1} = -z \sum \frac{1}{z^n} a^n$

$\| z^{1-n} a^n \|$