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Summary

- Spectrum and functional calculus
  - spectral radius
  - comparison of $\sigma_B(b) \supset \sigma_A(b)$

Recall

$f(z) = \sum_{n=0}^{\infty} c_n z^n$ holom. func. on $\mathbb{C}$

$\rightsquigarrow f(a) = \sum_{n=0}^{\infty} c_n a^n$ makes sense for

$\forall a \in A$ $A$: unital Banach alg.

Rem for fixed $a$ it's enough to assume

$\overline{\lim} \sqrt[n]{|c_n| \cdot \|a^n\|} \leq \overline{\lim} \sqrt[n]{|c_n|} \cdot \|a\| < 1.$

i.e. enough if $f(z)$ is defied on

$r \cdot \mathbb{D} = \{z : |z| < r\}$ for $r > \|a\|$.

$\rightsquigarrow \log(1-a) = -\sum_{n=1}^{\infty} \frac{1}{n} a^n$ makes sense

if $\|a\| < 1$, etc.

Prop. $f(z)$, $a \in A$ as above $\rightsquigarrow \sigma_A(f(a)) = f(\sigma_A(a))$

Step 1. We may assume $A$ is commutative

$\because$ $B_0 = \{$ polynomials in $a$, $(a-z)^{-1}$, $(f(a)-m)^{-1}$ $m \in \sigma_A(f(a))$, $z \in \rho_A(a)$ $\}$

$B$ = closure of $B_0$ comm.

$\rightsquigarrow \sigma_B(a) = \sigma_A(a)$, $f(a) \in B$, etc.

Step 2 $\sigma(f(a)) = f(\sigma(a))$

$\because$ $\sigma(f(a)) = \Gamma(f(a))(M_A)$ by Prop. (03.06)

$\left.\begin{array}{l} \Gamma(f(a))(M) = f(\Gamma(a))(M) \\ f(\Gamma(a))(M) = f(\Gamma(a)(M)) \end{array}\right)$ by def.

$\Gamma(a)(M) = \sigma(a)$ by Prop (03.06)

Spectral radius of $a \in \mathcal{A}$

$$r_{\mathcal{A}}(a) = \max_{z \in \sigma_{\mathcal{A}}(a)} |z|$$

Th'm (Beurling - Gelfand) $\mathcal{A}$ unital Banach alg

$a \in \mathcal{A} \rightsquigarrow r_A(a) = \lim_{n\to\infty} \sqrt[n]{\|a^n\|}$

Proof Step 1 We may assume $\mathcal{A}$ comm.

$B_0 = \{$ poly. in $a$, $(a^n - z)^{-1}$ $z \in \rho_{\mathcal{A}}(a^n)\}$

$B$ : clos. of $B_0$

Step 2 $r(a) \leq \underline{\lim}_{n\to\infty} \sqrt[n]{\|a^n\|}$ $\left(= \inf \sqrt[n]{\|a^n\|}\right)$

$\because$ Prop $\Rightarrow$ $\sigma(a)^n = \sigma(a^n) \Rightarrow r(a)^n = r(a^n) \leq \|a^n\|$

Step 3 $\exists\, C > 0$ s.t. $|z|^{1-n} \cdot \|a^n\| \leq C$ for $|z| > r(a)$

$\because$ Put $g(z) = (a-z)^{-1} = -\sum_{n=0}^{\infty} z^{1-n} a^n$

for $|z| > \|a\|$

$\forall \varphi \in \mathcal{A}^*$ we have $\varphi((a-z)^{-1}) = -\sum z^{1-n} \varphi(a^n)$

and this makes sense for $|z| > r(a)$

So $|z| > r(a) \Rightarrow |z|^{1-n} \cdot |\varphi(a^n)| \to 0$ $(n\to\infty)$

By Uniform Boundedness principle $\exists\, C > 0$

as in claim.

Step 4 $\overline{\lim}_{n\to\infty} \sqrt[n]{\|a^n\|} \leq r(a)$

$\because$ From step 3 $\sqrt[n]{\|a^n\|} \leq \sqrt[n]{|z|^{n-1} C} \to |z|$

for $|z| > r(a)$

So $\overline{\lim} \sqrt{\|a^n\|} \leq r(a)$. $\square$

Unif. Bdd princip.

$\mathcal{X}$ Ban. $(x_n)_{n=1}^{\infty}$ seq. in $\mathcal{X}$

$\forall \varphi \in \mathcal{X}^*$ $(\varphi(x_n))_{n=1}^{\infty}$ bdd in $\mathbb{C}$

$(\Leftarrow)\Rightarrow$ $(\|x_n\|)_{n=1}^{\infty}$ bdd

Comparison of $\sigma_{\mathcal{B}}(b)$ & $\sigma_{\mathcal{A}}(b)$ $b \in \mathcal{B} \subset \mathcal{A}$.

Ex. $\bar{\mathbb{D}} = \sigma_{\mathcal{B}}(z) \supset \mathbb{T} = \sigma_{C(\mathbb{T})}(z)$

$$\mathcal{B} =: \left\{ \sum_{n=0}^{\infty} c_n z^n \in C(\mathbb{T}) \right\}$$

Thm (Šilov) $b \in \mathcal{B} \subset \mathcal{A} \Rightarrow \partial \sigma_{\mathcal{B}}(b) \subset \partial \sigma_{\mathcal{A}}(b)$
boundary

Proof

Step 1 Enough to show $\partial \sigma_{\mathcal{B}}(b) \subset \sigma_{\mathcal{A}}(b)$

$\because$ $\sigma_{\mathcal{A}}(b) \subset \sigma_{\mathcal{B}}(b) \Rightarrow$ int. $\sigma_{\mathcal{A}}(b) \subset$ int. $\sigma_{\mathcal{B}}(b)$.

St. 2

Let $z \in \partial \sigma_{\mathcal{B}}(b)$, choose $\sigma_{\mathcal{B}}(b) \not\ni z_n \to z$

then $\|(a - z_n)^{-1}\| \to \infty$ $(n \to \infty)$

$\because$ We have $\|(a - z_n)^{-1}\| \geq \dfrac{1}{|z_n - z|}$

Otherwise $\|(a - z) - (a - z_n)\| < \dfrac{1}{\|(a - z_n)^{-1}\|}$

this implies $a - z$ is invertible in $\mathcal{B}$

(cf. 02.25)

Step 3 If $z \in \partial \sigma_{\mathcal{B}}(b)$ was not in $\sigma_{\mathcal{A}}(b)$

$\|(a - w)^{-1}\|$ would be bdd when $w \sim z$.

make sense, $\square$

Cor. Same setting:

Suppose $\rho_{\mathcal{A}}(b) = \coprod_{i=0}^{n} A_i$ $A_i$ conn.

$A_0$: unbounded "outside" region

Then $1 \leq j_1 < \cdots < j_m \leq n$ $\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b) \cup \left( \bigcup_{k=1}^{m} A_{j_k} \right)$

$(0 \leq m \leq n)$