Summary

- $L^\infty$ and maximal abelian subalgs of $\mathcal{L}(\mathcal{H})$
- $C^*$-algebras
  - permanence of spectrum

Normal operators from $L^\infty$.

$(X, \mu)$ measure space

$\rightsquigarrow$ Hilbert space $L^2(X,\mu) = \{ f : X \to \mathbb{C}$ meas, $\int |f(x)|^2 d\mu(x) < \infty \}$

Banach alg $L^\infty(X,\mu) = \{ f : X \to \mathbb{C}$ meas, bdd (outside a null set)$\}$

prod. $(f \cdot g)(x) = f(x) g(x)$

norm $\|f\|_\infty = \inf \{ t : \mu(\{x \in X : \underbrace{f(x)}_{\text{pt neglect } f(x) \le t \text{ a.e.}} > t\}) = 0 \}$

Prop. $f \in L^\infty(X, \mu) \rightsquigarrow T^f \in \mathcal{L}(L^2(X,\mu))$ by

$(T^f g)(x) = f(x) g(x)$ (we'll write $T^f = f$)

s.t. $\|T^f\| = \|f\|_\infty$.

Proof Step 1 $\|T^f\| \le \|f\|_\infty$.

$\because$ $g \in L^2(X,\mu) \rightsquigarrow \|fg\|_2 \le \|f\|_\infty \cdot \|g\|_2$. by

$$\int |f(x) g(x)|^2 d\mu(x) \le \int |t \cdot g(x)|^2 d\mu(x) = t^2 \|g\|_2^2$$

whenever $|f(x)| \le t$ a.e.

Step 2 $\|T^f\| \ge \|f\|_\infty$

$A_n = \{ x : |f(x)|^2 > \|f\|_\infty^2 - \frac{1}{n} \}$

has pos. measure. $\chi_{A_n}(x) = \begin{cases} 1 & x \in A_n \\ 0 & x \notin A_n \end{cases}$

$\rightsquigarrow \|f \chi_{A_n}\|_2^2 = \int_{A_n} \underbrace{|f(x)|^2}_{\ge \|f\|_\infty^2 - \frac{1}{n}} d\mu(x) \ge (\|f\|_\infty^2 - \frac{1}{n}) \cdot \|\chi_{A_n}\|_2^2$

$\|\chi_{A_n}\|_2^2 = \mu(A_n) > 0 \Rightarrow \|T^f\| \ge \|f\|_\infty - \frac{1}{n}$.

- $(f_1 f_2) g = f_1 (f_2 g) \rightsquigarrow$ prod. in $L^\infty(X,\mu)$ becomes prod. in $\mathcal{L}(L^2(X,\mu))$

- $(fg, h)_{L^2} = (g, \bar{f} h)_{L^2} \rightsquigarrow \underset{\text{adjoint op.}}{f^*} = \underset{\text{ptwise conjugation.}}{\bar{f}}$

$\rightsquigarrow L^\infty(X,\mu)$ is a (closed) subalg of $\mathcal{L}(L^2(X,\mu))$ closed under op. $T \mapsto T^*$
(<u>self adjoint</u> subalgebra)

later? x ? Motto: self adjoint subalgs of $\mathcal{L}(\mathcal{H})$ behave nicely for spectral theory

<u>Maximal abelian subalgebra</u> $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$

- comm subalg, (sometimes selfadj, also)
- if $T \in \mathcal{L}(\mathcal{H})$ satisfies $\forall S \in \mathcal{A}$ $TS = ST$
  then $T \in \mathcal{A}$. (write $\underset{\{\text{set of such } T\}}{\mathcal{A}'} = \mathcal{A}$)

Prop. $L^\infty(X,\mu) \subset \mathcal{L}(L^2(X,\mu))$ is a MASA if $(X,\mu)$ is a (union of) fin. meas sp. $\mu(X) < \infty$

Proof. (We assume $\mu(X) < \infty \Rightarrow 1 \in L^2(X,\mu)$)

Take $T \in L^\infty(X,\mu)'$, put $f = T.1$

Step 1 $\|f\|_\infty \le \|T\|$ $(\Rightarrow f \in L^\infty(X,\mu))$

$\because$ Put $A_n = \{x \in X : |f(x)|^2 \ge \|T\|^2 + \frac{1}{n}\}$

$$f \chi_{A_n} = \chi_{A_n} f = \chi_{A_n} T.1 \underset{T \in L^\infty(X,\mu)'}{=} T \chi_{A_n}$$

$$\rightsquigarrow \int_{A_n} |f|^2 d\mu = \|T \chi_{A_n}\|_2^2 \le \|T\|^2 \cdot \|\chi_{A_n}\|_2^2$$

LHS $\ge (\|T\|^2 + \frac{1}{n}) \mu(A_n)$

$\mu(A_n) = \|\chi_{A_n}\|_2^2 \Rightarrow \mu(A_n)$ must be 0.

Prop. $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ max. abelian, $T \in \mathcal{A}$

$\sigma_{\mathcal{A}}(T) = \sigma_{\mathcal{L}(\mathcal{H})}(T)$

Proof. Step 1 $z \in \rho_{\mathcal{L}(\mathcal{H})}(T)$, $S \in \mathcal{A}$

$\Rightarrow (T-z)^{-1}$ and $S$ commute

$\because$ We have $(T-z)S = S(T-z)$

mult. by $(T-z)^{-1}$ from both sides

$S(T-z)^{-1} = (T-z)^{-1}S$

Step 2 claim.

$(T-z)^{-1} \underset{\text{Step 1}}{\in} \mathcal{A}' \underset{\text{max ab.}}{=} \mathcal{A}$ for $z \in \rho_{\mathcal{L}(\mathcal{H})}(T)$

$\Rightarrow \rho_{\mathcal{L}(\mathcal{H})}(T) = \rho_{\mathcal{A}}(T)$. ☺

Motto : - - -

$C^*$-algebras

(Banach) $\underline{*\text{-algebra}}$ : (Banach) alg. $\mathcal{A}$

with map $\mathcal{A} \to \mathcal{A}$, $T \mapsto T^*$ s.t.

- $(\lambda T + S)^* = \bar{\lambda} T^* + S^*$ conj. linear.
- $T^{**} = T$ involutive
- $(ST)^* = T^* S^*$ antimulitiplicative.

$\underline{C^*\text{-algebra}}$ : Banach $*$-alg s.t.

$\|T\|^2 = \|T^*T\| \quad (= \|TT^*\|)$

Rem $\|T^*\| = \|T\|$ from $\|TT^*\| \le \|T\| \cdot \|T^*\|$

Prop. $\mathcal{L}(\mathcal{H})$ is a $C^*$-alg.

Proof Step 1 $\|T^*\| = \|T\|$

$\because$ $\|T\| = \sup_{\|u\| = 1 = \|v\|} \underbrace{|(Tu, v)|}_{= |(T^* v, u)|}$

St. 2 $\|T^*T\| \le \|T\|^2$

Step 3 $\|T^*T\| \ge \|T\|^2$

$\because$ fix $\varepsilon$, pick $\|u\| = 1$, $\|Tu\|^2 > \|T\|^2 - \varepsilon$

$\|T^*T\| \ge |(T^*Tu, u)| = \|Tu\|^2$

Rem. $\mathcal{A} \subset \mathcal{L}(H)$ selfadj closed subalg

$\Rightarrow$ $\mathcal{A}$ is also a $C^*$-alg.

In fact any $C^*$-alg is isom to such one.

Ex. $C(X)$, $L^\infty(X, \mu)$. $C^*$-alg.

Thm. $\mathcal{A}$ $C^*$-alg, $\mathcal{B} \subset \mathcal{A}$ $C^*$-subalg.

(closed selfadj alg)

$b \in \mathcal{B} \Rightarrow \sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b)$

Proof. St. 1 Enough to show $\exists b^{-1} \in \mathcal{A} \to b^{-1} \in \mathcal{B}$

$\because$ This will imply $z \in \rho_{\mathcal{A}}(b) \Rightarrow (b - z)^{-1} \in \mathcal{B}$

$\Rightarrow z \in \rho_{\mathcal{B}}(b)$

St. 2 $\exists b^{-1} \Rightarrow \exists (b^* b)^{-1}$

$\because$ $(b^* b)^{-1} = b^{-1} (b^{-1})^*$ from $(xy)^* = y^* x^*$

$1^* = 1$.

St. 3 $\sigma_{\mathcal{B}}(b^* b) \subset \mathbb{R}$

$\because$ $u = \exp(\sqrt{-1}\, \underbrace{b^* b}_{\text{s.a.}}) \in \mathcal{B}$ is unitary!

$u^* = u^{-1}$.

$\|u\| = \sqrt{\|u^* u\|} = 1$ from $C^*$-cond.

same for $\|u^*\|$

$\|u\| = 1 = \|u^{-1}\| \Rightarrow \sigma_{\mathcal{B}}(u) \subset \mathbb{T}$

(from $z \in \rho(u) \Leftrightarrow z^{-1} \in \rho(u^{-1})$

$(u^{-1} - z^{-1})^{-1} = -zu(u - z)^{-1}$

so $\exp(\sqrt{-1}\, \sigma_{\mathcal{B}}(b^* b)) \subset \mathbb{T}$.

St. 4 $\sigma_{\mathcal{A}}(b^* b) = \sigma_{\mathcal{B}}(b^* b)$

$\because$ Šilov's thm. & $\sigma_{\mathcal{B}}(b^* b) \underset{\text{St 3}}{=} \partial \sigma_{\mathcal{B}}(b^* b)$

St. 5 $\exists (b^* b)^{-1} \in \mathcal{A} \Rightarrow \exists (b^* b)^{-1} \in \mathcal{B}$.

$\because$ same as $0 \notin \sigma(b^* b)$