Summary

- Permanence of spectrum in C^*-subalg.
- Gelfand - Naimark theorem

"Th'm" from last note.

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Gelfand - Naimark theorem

"commutative C^*-alg is always $C(X)$".

$\rightsquigarrow$ What can we do in $C(X)$?

- spectrum = range as function

$\sigma(f) = \{f(x) ; x \in X\}$

- $\left.\begin{array}{l} g(z) \quad z \in \mathbb{C} \quad (\text{or } z \in \sigma(f)) \\ f(x) \quad x \in X \end{array}\right\}$ continuous

$\Rightarrow g(f(x))$ also continuous.

replace $f(x)$ by $a \in \mathcal{A}$

$\rightsquigarrow$ "substitute z by a" $g(a)$

<u>continuous</u> <u>functional</u> <u>calculus</u>

e.g. $g(t) = \sqrt{t}$ on $\mathbb{R}$, $\sigma(a) \subset \mathbb{R}$

$\rightsquigarrow$ "$\sqrt{a}$" s.t. $\sqrt{a}^2 = a$

Th'm (Gelfand - Naimark)

$\mathcal{A}$: commutative unital C^*-alg.

$\rightsquigarrow$ The Gelfand transform $\Gamma : \mathcal{A} \to C(M_{\mathcal{A}})$ is isom of C^*-alg's.

i.e. $\underbrace{\|a\| = \max_{\varphi \in M_{\mathcal{A}}} |\varphi(a)|}_{\Gamma(a)(\varphi)}$, $\underbrace{\varphi(a^*)}_{\Gamma(a^*)(\varphi)} = \underbrace{\overline{\varphi(a)}}_{\Gamma(a)^*(\varphi)}$, etc.

Proof Step 1 $\varphi \in M_{\mathcal{A}}$, $a \in \mathcal{A} \Rightarrow \varphi(a^*) = \overline{\varphi(a)}$

$\because$ We use $\sigma_{\mathcal{A}}(b) = \sigma_{C(M_{\mathcal{A}})}(\Gamma(b))$ Mar. 4 "Prop"

and $b = b^* \Rightarrow \sigma_{\mathcal{A}}(b) \subset \mathbb{R}$ Th'm Step 3

$$a = \underbrace{\tfrac{1}{2}(a + a^*)}_{\text{"Re } a\text{"}} + \sqrt{-1}\,\Big(\underbrace{\tfrac{1}{2\sqrt{-1}}(a - a^*)}_{\text{"Im } a\text{"}}\Big)$$

Re a, Im a are selfadj

$\rightsquigarrow$ $\Gamma(\mathrm{Re}\, a)(\varphi)$, $\Gamma(\mathrm{Im}\, a)(\varphi) \in \mathbb{R}$

$a = \mathrm{Re}\, a + \sqrt{-1}\,\mathrm{Im}\, a$, $a^* = \mathrm{Re}\, a - \sqrt{-1}\,\mathrm{Im}\, a$

$$\Rightarrow \Gamma(a)(\varphi) = \Gamma(\mathrm{Re}\, a)(\varphi) + \sqrt{-1}\,\Gamma(\mathrm{Im}\, a)(\varphi)$$
$$= \overline{\Gamma(a^*)(\varphi)}$$

Step 2 $\|a\|_{\mathcal{A}} = \|\Gamma(a)\|_{C(M_{\mathcal{A}})}$.

$\because$ $\|a\|^2 \underset{C^*\text{-alg}}{=} \|a^* a\| \underset{C^*\text{-alg}}{=} \|(a^* a)^2\|^{\frac{1}{2}}$

$$\underset{\text{induction}}{=} \|(a^* a)^{2^k}\|^{\frac{1}{2^k}}$$

$$\lim \|(a^* a)^{2^k}\|^{\frac{1}{2^k}} = \text{spec. rad. of } a^* a$$

$$= \max_{z \in \sigma(a^* a)} |z| \underset{\text{Mar 4 Prop}}{=} \|\Gamma(a^* a)\|$$

By Step 1 $\Gamma(a^* a) = \Gamma(a)^* \Gamma(a)$

$C(M_{\mathcal{A}})$ is a C^*-alg. $\|\Gamma(a)^* \Gamma(a)\| = \|\Gamma(a)\|^2$.

Step 3 Γ is an iso. of algs.

$\because$ • By Step 2 Ran Γ is a closed subsp. of $C(M_{\mathcal{A}})$

• Γ is an alg hom. (inj. by Step 2)

$\rightsquigarrow$ enough to prove Ran Γ is dense in $C(M_{\mathcal{A}})$

use Stone-Weierstrass

Ran(Γ) is a selfadj subalg of $C(M_{\mathcal{A}})$

separate pts : $\varphi \neq \psi \Rightarrow \exists f \in \mathrm{Ran}\,\Gamma$ s.t. $f(\varphi) \neq f(\psi)$

if $\Gamma(a)(\varphi) = \Gamma(a)(\psi)$ for all $a \in \mathcal{A}$, $\varphi = \psi$ ◻

Spectral theorem) $\mathcal{H}$: Hilbert space

$T \in \mathcal{L}(\mathcal{H})$ normal operator ($T^*T = TT^*$)

$C^*(T)$: C^*-alg. generated by T

norm closure of {polynoms in T and T^*}

$\Rightarrow C^*(T) \simeq C(\sigma(T))$

$T \longleftrightarrow$ coord func. $f(z) = z$.

Proof. T normal $\Rightarrow C^*(T)$ commutative

so we already know $C^*(T) \simeq C(M_{C^*(T)})$

We want homeo $\theta : M_{C^*(T)} \to \sigma(T)$.

Put $\theta(\varphi) = \varphi(T)$

St. 1 θ is surjective

$\because$ $\sigma(T) = \mathrm{Ran}\, \Gamma(T) = \{\varphi(T) : \varphi \in M_{C^*(T)}\}$

St. 2. θ is injective

$\because$ If $\theta(\varphi) = \theta(\psi)$,

- $\varphi(T^n) = \varphi(T)^n = \psi(T)^n = \psi(T^n)$
- $\varphi(T^*) = \Gamma(T^*)(\varphi) = \Gamma(T)^*(\varphi) = \overline{\varphi(T)}$

$= \overline{\psi(T)} = \psi(T^*)$

$\rightsquigarrow \varphi(T^m T^{*n}) = \psi(T^m T^{*n})$

$\rightsquigarrow$ φ and ψ agree on polynoms of T & T^*

$\xrightarrow[\text{density}]{}$ agree on $C^*(T)$, i.e. $\varphi = \psi$

St. 3 Claim of thm

$\because$ θ is homeo by $M_{C^*(T)}$ cpt, $\sigma(T)$ Hausdorff

$f(z) = z \rightsquigarrow \Gamma(T)(\varphi) = \theta(\varphi) = f(\theta(\varphi))$

So $\Gamma(T) \longleftrightarrow f(z)$

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