2019.03.20

Summary

- Stone-Weierstrass th'm
- Examples for spectral th'm
- polar decomposition.

Th'm (Stone-Weierstrass)

X : cpt (Hausdorff) top. sp.

$\mathcal{A} \subset C(X)$ unital subalg s.t.

- $f \in \mathcal{A} \Rightarrow f^* = \bar{f} \in \mathcal{A}$ (selfadj as subsp.)
- $x, y \in X,\ x \neq y \Rightarrow \exists f \in \mathcal{A}$ $f(x) \neq f(y)$

(separates points)

Then $\mathcal{A}$ is dense in $C(X)$.

Rem. We need both conds : $\mathbb{T} = X$

$\mathcal{A} = \{ f(z) \in C(\mathbb{T}) : f(z) = \sum_{n=0}^{\infty} c_n z^n \}$ not dense $\because$ not closed under $f \mapsto f^*$

$\mathcal{A} = \{ f(z) : f(z) = f(\bar{z}) \}$ not dense

$(\operatorname{dist}(z, \mathcal{A}) \geq 1)$ $\because$ $\sqrt{-1}, -\sqrt{-1}$ cannot be separated.

Proof Put $\mathcal{A}_{\mathbb{R}} = \mathcal{A} \cap C(X; \mathbb{R})$.

Step 1. Enough to show $\mathcal{A}_{\mathbb{R}} \subset C(X; \mathbb{R})$ dense

$\because$ $f = \operatorname{Re} f + \sqrt{-1} \operatorname{Im} f$, $g, h \in \mathcal{A}_{\mathbb{R}}$ close to $\operatorname{Re} f$ & $\operatorname{Im} f$ $\Rightarrow$ $g + \sqrt{-1} h$ close to f.

Step 2 $f \in \bar{\mathcal{A}} \Rightarrow |f| \in \bar{\mathcal{A}}_{\mathbb{R}}$

$\because$ By rescaling we may assume $\|f\| = 1$.

$\sqrt{1-t} = \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} t^n$ rad. conv. 1.

$|f|^2 = f^* f \in \bar{\mathcal{A}}$, $0 \leq 1 - |f| \leq 1$.

$|f| = \sqrt{1 - (1 - |f|^2)} = \sum (-1)^n \binom{1/2}{n} (1 - |f|^2)^n$.

Given $\varepsilon > 0$ we can find $M \gg 1$

s.t. $\| g_\delta^{\frac{1}{2}} - \sum_{n=0}^{M} (-1)^n \binom{1/2}{n} (1 - g_\delta)^n \| < \varepsilon$

put $g_\delta = \delta + (1-\delta)|f|^2 \in \bar{A}$.

$0 \le 1 - g_\delta \le 1 - \delta$, $\| g_\delta^{\frac{1}{2}} - |f| \|$ small when δ small

$\rightsquigarrow$ we could approx. $|f|$ by an elem in $\bar{A}$.

Step 3. $f, g \in \bar{A}_{\mathbb{R}} \Rightarrow (f \vee g)(x) = \max(f(x), g(x))$

$(f \wedge g)(x) = \min(f(x), g(x))$

are in $\bar{A}_{\mathbb{R}}$.

$\because$ $f \vee g = \frac{1}{2}(f + g + |f - g|)$

and use Step 2

Step 4 $x \neq y \in X$, $a, b \in \mathbb{R}$

$\Rightarrow \exists f \in \bar{A}_{\mathbb{R}}$ s.t. $f(x) = a$, $f(y) = b$.

$\because$ take $g \in A$ s.t. $g(x) \neq g(y)$

looking at $\mathrm{Re}\, g$, $\mathrm{Im}\, g$, WMA $g \in \bar{A}_{\mathbb{R}}$

$f(z) = a + (b-a)\dfrac{g(z) - g(x)}{g(y) - g(x)}$ does the job

Step 5 Fix $y \in X$, $f \in C(X; \mathbb{R})$

$\forall \varepsilon > 0$ $\exists g_y \in \bar{A}_{\mathbb{R}}$ s.t.

$g_y(y) = f(y)$, $g_y(x) \le f(x) + \varepsilon$ for $x \in X$

$\because$ For each x choose $h_x \in \bar{A}_{\mathbb{R}}$

$h_x(y) = f(y)$, $h_x(x) = f(x)$ (by Step 4)

$U_x = \{ z \in X : h_x(z) < f(z) + \varepsilon \}$ is an open neigh. of x.

By compactness $\exists\, x_1, \dots, x_n \in X$ s.t.

$X = \bigcup_{i=1}^{n} U_{x_i} \rightsquigarrow g_y = h_{x_1} \wedge \dots \wedge h_{x_n} \in \mathcal{A}$ does the job (Step 3).

Step 6. $f \in C(X;\mathbb{R})$, $\varepsilon > 0$

$\exists\, g \in \mathcal{A}$ $\quad f - \varepsilon \le g \le f + \varepsilon.$ $(\Rightarrow \|f - g\| \le \varepsilon)$

$\because$ $V_y = \{x \in X : g_y(x) > f(x) - \varepsilon\}$

is open neigh. of y.

By cpt-ness $\exists\, y_1, \dots, y_m$ s.t. $X = \bigcup_{i=1}^{m} V_{y_i}$

$g = g_{y_1} \vee \dots \vee g_{y_m} \in \mathcal{A}$ does the job
Step 3 ☒

Example of spectral th'm (diagonalizable ops)

1. $T \in M_n(\mathbb{C})$ normal matrix

$\rightsquigarrow$ $\exists$ unitary $U \in M_n(\mathbb{C})$ s.t. $T = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^*$

$\sigma(T) = \{\lambda_1, \dots, \lambda_n\}$. any cont. func.

on $\sigma(T)$ can be represented by a

polynomial. $f(z) = \sum_{k=0}^{n-1} c_k z^k$

$f(T) = U \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} U^*$.

(e.g. $\left(U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^*\right)^2 = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \underbrace{U^* U}_{I_n} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^*$

$= U \begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix} U^*$)

2. $T \in \mathcal{L}(\mathcal{H})$ has ONB of eigenvectors $(u_n)_{n=1}^{\infty}$

$T u_n = \lambda_n u_n$.

$\rightsquigarrow$ $U : \ell_2 \mathbb{N} \to \mathcal{H}$, $\delta_n \mapsto u_n$ is a unitary op.

$T = U (\lambda_n)_{n=1}^{\infty} U^*$. $\quad (\lambda_n)_n \in \ell_\infty \mathbb{N}$.

$f(T) = U \, (f(\lambda_n))_n \, U^*$.

<u>Polar decomposition</u> $T = VA$ ← positive
$\uparrow$ partial isomet

if V was unitary: $T^*T = AV^*VA = A^2$

so A would be $\sqrt{T^*T}$ ← makes sense by $\sigma(T^*T) \subset [0, \infty)$.

Th'm. $\forall T \in \mathcal{L}(\mathcal{H})$ $\exists V, A$ as above.

They are unique if we require $\operatorname{Ker} A = \operatorname{Ker} V$

Proof. Existence: take $A = \sqrt{T^*T}$

Step 1 $\|Au\| = \|Tu\|$ for $u \in \mathcal{H}$

$\because$ $\|Au\|^2 = (Au, Au) \underset{A=A^*}{=} (A^2u, u) = (T^*Tu, u)$
$= \|Tu\|^2$

Step 2 $V: \operatorname{Ran} A \oplus (\operatorname{Ran} A)^\perp \to \mathcal{H}$, $Av + w \mapsto Tv$

extends to a partial isomet.

(in a unique way)

$\because$ well-def: Step 1

unique: $\operatorname{Ran} A \oplus (\operatorname{Ran} A)^\perp \subset \mathcal{H}$ dense

Step 3 $T = VA$. from def.

Uniqueness Suppose $T = \underset{\text{p.iso}}{W}\underset{\text{pos}}{B}$, $\operatorname{Ker} B = \operatorname{Ker} W$

Step 4 W^*W is prj onto $\overline{\operatorname{Ran} B}$

$\because$ W^*W is prj onto $(\operatorname{Ker} W)^\perp = (\operatorname{Ker} B)^\perp$
$\underset{\text{Feb 20 Prop}}{=} \overline{\operatorname{Ran} B^{(*)}}$

Step 5 $A = B$

$\because$ $A^2 = T^*T = BW^*WB \underset{\text{Step 4}}{=} B^2$.

Step 6 $V = W$.

$\because$ $VA = WA \Rightarrow V = W$ on $\operatorname{Ran} A$. $V = 0 = W$ on compl.