Summary
- eigenvector from isolated spectrum.
- towards Borel functional calculus
  - strong / weak operator topology on $\mathcal{L}(\mathcal{H})$

Eigenvector from isolated spectrum

$T \in \mathcal{L}(\mathcal{H})$ normal,

λ : isolated point of $\sigma(T)$, i.e.:

$\exists$ neigh. U of λ $U \cap \sigma(T) = \{\lambda\}$

$\chi_{\{\lambda\}}(z) = \begin{cases} 1 & (z = \lambda) \\ 0 \end{cases}$ is cont. on $\sigma(T)$

We have alg. iso $C^*(T) \simeq C(\sigma(T))$

so we have $P_\lambda = \chi_{\{\lambda\}}(T) \in C^*(T)$

$\chi_{\{\lambda\}}^2 = \chi_{\{\lambda\}} = \chi_{\{\lambda\}}^*$ in $C(\sigma(T))$

$\rightsquigarrow P_\lambda$ is the same, proj on $\mathcal{H}$.

Prop 1) $P_\lambda \neq 0$

2) $\mathrm{Ran}\, P_\lambda = \{u \in \mathcal{H} : Tu = \lambda u\}$.

Proof S.1 2) "$\subset$": $f(z) = z$ satisfies $f(T) = T$.

$\because f \cdot \chi_{\{\lambda\}} = \lambda \chi_{\{\lambda\}} \Rightarrow T \chi_{\{\lambda\}}(T) = \lambda \chi_{\{\lambda\}}(T)$

so $T \chi_{\{\lambda\}}(T) u = \lambda \chi_{\{\lambda\}}(T) u$

S.2 If "" $\varepsilon > 0$, $A = \{z : |z - \lambda| > \varepsilon\}$

satisfies $A \cup \{\lambda\} \supset \sigma(T)$,

$\chi_A(T) + \chi_{\{\lambda\}}(T) = 1 \quad (= \mathrm{Id}_{\mathcal{H}})$

$\|(T \cdot \chi_A(T) - \lambda \chi_A(T)) \cdot g(T) = \chi_A(T)$ for $\exists g(z) \in C(\sigma(T))$

$\because$ corresp. things happen in $C(\sigma(T))$

S. 3. 1))

$\because$ $\chi_{\{\lambda\}}(T) = 0 \Leftrightarrow \chi_A(T) = 1$

If this happened, $T - \lambda$ is invertible

$\Rightarrow \lambda \notin \sigma(T)$.

S. 4 2) "$\supset$" or $\{Tu = \lambda u\} \ominus \operatorname{Ran} \chi_{\{\lambda\}}(T) = 0$

$\because$ On $\operatorname{Ran} \chi_A(T) = (\operatorname{Ran} \chi_{\{\lambda\}}(T))^{\perp}$

$T - \lambda$ is invertible $\Rightarrow Tu \neq \lambda u$

for vecs in this sp. $\square$

Ex. $T \in M_n(\mathbb{C})$, normal

$\sigma(T) = \{\lambda_1, \dots, \lambda_k\}$ ($k \le n$, all distinct)

$e_i(z) = \prod_{j \neq i} \frac{(z - \lambda_j)}{(\lambda_i - \lambda_j)}$ $\qquad e_i(z) = \chi_{\{\lambda_i\}}(z)$ on $\sigma(T)$.

$e_i(T)$ proj. onto λ_i - eigensp. of T

Towards Borel functional calculus

Generally, $A \subset \mathbb{R}$ Borel measurable

$\chi_A(z)$ for or $f(z)$ meas

T normal $\rightsquigarrow$ we want to make sense of $\chi_A(T)$, $f(T)$

$\rightsquigarrow$ we get projection valued measure

$A \mapsto E(A) = \chi_A(T)$

we can formally write $f(T) = \int f(z)\, dE(z)$, etc

$u, v \in \mathcal{H} \rightsquigarrow$ usual cplx meas.

$\mu_{u,v}(A) = (E(A)u, v)$ s.t.

$(f(T)u, v) = \int f(z)\, d\mu_{u,v}(z)$.

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$f(t)$ bdd meas in $0 \le t \le 1$

but not cont $\Rightarrow f \notin C([0,1])$, but

$\exists$ seq $f_n(t) \in C([0,1])$ s.t.

$f_n(t) \to f(t)$ for each t

$|f_n(t)| \le \|f\|_\infty \rightsquigarrow (f_n g, h) \to (fg, h)$

for $g, h \in L^2([0,1])$ (Dominated conv.)

and $\|(f - f_n) g\|_2 \to 0$

So we should look at pointwise

convergence as <u>operators</u>

Def. <u>Strong</u> <u>operator</u> <u>topology</u> on $\mathcal{L}(\mathcal{H})$

top. of ptwise conv. as maps

from $(\mathcal{H}$, norm top) to $(\mathcal{H}$, norm top.)

<u>Weak</u> <u>op.</u> <u>top.</u> on $\mathcal{L}(\mathcal{H})$:

top of ptwise conv. as maps

from $(\mathcal{H}$, weak top) to $(\mathcal{H}$, weak top)

So $(T_i)_i$ seq / net in $\mathcal{L}(\mathcal{H})$ conv. to T

- in str. op. top.: $\forall u \in \mathcal{H}$ $\|T_i u - Tu\| \to 0$
- in wk. op. top.: $\forall u, v \in \mathcal{H}$ $(T_i u, v) \to (Tu, v)$

wk op top. $\subset$ str op. top. $\subset$ norm top. (str. top.)

wk op top. $\subset$ wk top as Ban. $\subset$ norm top.

Base of open sets:

$\mathcal{O}_{u,v,\varepsilon} = \{ T : \|Tu - v\| < \varepsilon \}$ str. op.

$\mathcal{O}_{u,v,\lambda,\varepsilon} = \{ T : |(Tu, v) - \lambda| < \varepsilon \}$ wk. op.

Ex. U unilat shift on $\ell_2 \mathbb{N}$.

$U^n \to 0$ in wk op. top. $(U^n a, b) \to 0$

$\not\to 0$ in str. op. top. $\|U^n a\| \not\to 0$

Prop. Following maps are cont. in str op. (1, 2, 2′) and wk op. (1, 2, 2′, 3) topologies.

1) $\alpha: \mathcal{L}(\mathcal{H}) \times \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $(S, T) \mapsto S+T$.

2) $\lambda_A: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $T \mapsto AT$ for fixed A

2′) ρ_A: ~ $T \mapsto TA$

3) $j: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $T \mapsto T^*$ only for wk.

Pf. We show the inv. img of prev. bases are open. we check str. ver for 1, 2, wk for 3.

1) suppose $S+T \in \mathcal{O}_{u,v,\varepsilon}$ $(\|(S+T)u - v\| < \varepsilon)$

put $\delta = \frac{1}{2}(\varepsilon - \|(S+T)u - v\|)$, $w = Su$, $w' = Tu$

if $S' \in \mathcal{O}_{u,w,\delta}$, $T' \in \mathcal{O}_{u,w',\delta}$

$\|(S'+T')u - (S+T)u\| < \varepsilon - \|(S+T)u - w\|$

so $S'+T' \in \mathcal{O}_{u,v,\varepsilon}$.

$\mathcal{O}_{u,w,\delta} \times \mathcal{O}_{u,w',\delta}$ is $\left(\begin{array}{l}\text{open neigh of } (S,T) \\ \text{contained in } \alpha^{-1}(\mathcal{O}_{u,v,\varepsilon})\end{array}\right.$

2) suppose $AT \in \mathcal{O}_{u,v,\varepsilon}$ $\|ATu - v\| < \varepsilon$.

put $\delta = \frac{1}{\|A\|}(\varepsilon - \|A \cdot Tu - v\|)$, $w = Tu$.

If $T' \in \mathcal{O}_{u,w,\delta}$ $\|AT'u - ATu\| \le \|A\| \cdot \|T'u - w\|$

$< \varepsilon - \|ATu - v\| \Rightarrow AT' \in \mathcal{O}_{u,v,\varepsilon}$

2′) similar.

3) $j^{-1}(\mathcal{O}_{u,v,\lambda,\varepsilon}) = \mathcal{O}_{v,u,\bar\lambda,\varepsilon}$ by $|(Tu,v) - \lambda| = |(T^*v,u) - \bar\lambda|$