Summary
- Weak op. top. on $L^\infty(X,\mu)$
- Cyclic vector.
  - Measure from inner product.
  - Borel functional calculus

Weak op. top. on L^∞

Prop. (X,μ) measure s.p.
restriction of wk op. top. on
$L^\infty(X,\mu) \subset \mathcal{L}(L^2(X,\mu)) = \sigma(L^\infty(X,\mu), L^1(X,\mu))$ topology.

Call this $\longleftarrow$ ("the" w^*-top. on L^∞)

Pf. wk op. top. : open subbase was

$\mathcal{O}_{u,v,\lambda,\varepsilon} = \{T : |(Tu,v) - \lambda| < \varepsilon\}$

$\sigma(L^\infty(X,\mu), L^1(X,\mu))$-top : open subbase

$\mathcal{O}_{m,A} : \{f \in L^\infty : \underbrace{(f,m)}_{= \int_{x\in X} f(x) m(x) d\mu(x)} \in A\}$

$m \in L^1$, $A \underset{\text{opn}}{\subset} \mathbb{C}$

Enough to consider $A_{\lambda,\varepsilon} = \{z : |z-\lambda| < \varepsilon\}$

(or take $\mathcal{O}_{u,B} = \{T : Tu \in B\}$
$B \subset \mathcal{H}$ wkly open

enough to use $B_{v,A} = \{u' : (u',v) \in A\}$).

$m \in L^1(X,\mu)$ can be written as $m = u\bar{v}$
for $u, v \in L^2$.

$\rightsquigarrow \mathcal{O}_{m,A_{\lambda,\varepsilon}} = \mathcal{O}_{u,v,\lambda,\varepsilon} \cap L^\infty(X,\mu)$ ⊠

Th'm 1 X : cpt top. sp.

μ : fin. (pos.) regular Borel measure

$C(X)_1 \subset L^\infty(X,\mu)_1$ w^*-dense.

Proof Step 1 (simple) step funcs. are w^*-dense

$\because$ $f \in L^\infty(X,\mu) \Rightarrow \exists f_n$ step funcs $\|f_n\|_\infty \le \|f\|$,

$f_n(x) \to f(x)$ a.e.

Dominated conv. for $f_n u \bar{v}$

$\rightsquigarrow f_n \to f$ in w-op. top

Step 2. step funcs in $L^\infty(X,\mu)_1 \in w^*$-clos of $C(X)_1$

$\because$ $f(x) = \sum_{i=1}^n \lambda_i \chi_{A_i}(x)$ $|\lambda_i| \le 1$, $(A_i)_{i=1}^n$

s.t. $X = \coprod A_i$ (we allow $\lambda_i = 0$)

$\exists K_i \subset A_i$ cpt $\sum \nu(A_i \setminus K_i) < \frac{\varepsilon}{2}$

Tietze extension th'm : $\exists g \in C(X)$ s.t.

- $g(x) = \lambda_i$ $(x \in K_i)$, • $\|g\|_\infty \le 1$.

(Fix $h \in L^1(X,\mu)$, $\varepsilon > 0$) any such g sat.

$$\int \underbrace{(f-g)}_{\substack{0 \text{ on } K_i\ \forall i \\ \text{at most } 2}}(x)h(x)d\mu(x) = \sum_{i=1}^n \int_{A_i \setminus K_i} (f-g)\cdot h\, d\mu$$

$\rightsquigarrow$ abs. val $\le \sum_{i=1}^n 2\int_{A_i \setminus K_i} |h|\, d\mu < \varepsilon$

Since $\nu(B) = \int_B |h|(x)\, d\mu(x)$ is also regular

Cor. $C(X) \subset L^\infty(X,\mu)$ wk. op. dense

as subsp. of $\mathcal{L}(L^2(X,\mu))$

<u>Cyclic vector</u> for $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$

$v \in \mathcal{H}$ s.t. $\mathcal{A}v = \{Tv : T \in \mathcal{A}\}$ dense in $\mathcal{H}$

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Th'm 2 $T \in \mathcal{L}(\mathcal{H})$ normal, $v \in \mathcal{H}$ cyclic vector for $C^*(T)$.

$\Rightarrow \exists$ finite reg. Borel meas. μ_v on $\sigma(T)$

unitary $U : \mathcal{H} \to L^2(\sigma(T), \mu_v)$ s.t.

$U^* L^\infty(\sigma(T), \mu) U$ = weak op. top. closure of $C^*(T)$ $(=: W^*(T))$

Proof. Step 1 construction of μ_v

Define $\omega \in C(\sigma(T))^*$ by $\omega(f) = (f(T)v, v)$.

$\omega(f) \geq 0$ for $f \geq 0$

$|\omega(f)| \leq \|f\|_\infty \cdot \|v\|^2$

bounded pos. functional

$\underset{\text{Riesz rep.}}{\Longrightarrow}$ $\exists \mu_v$ s.t. $\omega(f) = \int f(x) d\mu_v(x)$.

Step 2 Const. of U

On $C^*(T)v \subset \mathcal{H}$ def. U by

$U(f(T)v) = f \in C(\sigma(T)) \subset L^2(\sigma(T), \mu_v)$.

$\|U f(T) v\|^2 = \|f\|_{L^2(\mu_v)} = (|f|^2(T) v, v)$

$= (f(T)v, f(T)v) = \|f(T)v\|^2$

so U is isometry $\to$ extends to $\mathcal{H}$

Range U is dense $C(\sigma(T)) \underset{\text{dense}}{\subset} L^2(\sigma(T), \mu_v)$.

Step 3 $U^* L^\infty(\sigma(T), \mu_v) U = W^*(T)$

$\because$ $U^* C(\sigma(T)) U = C^*(T)$

(check on $C^*(T)v$: $f, g \in C(\sigma(T))$

$U^* f U g(T) v = U^{-1} f g = (fg)(T) v$

$= f(T) g(T) v$.

so $C^*(T)$ and $C(\sigma(T))$ are unitarily equiv.

take wk op. clos. $\rightsquigarrow$ $W^*(T)$ & $L^\infty(\sigma(T), \mu_v)$

Borel func. calc. for $T \in \mathcal{L}(\mathcal{H})$ normal

We know: if $C^*(T)$ has a cyclic vec,

$f(x)$: bdd (Borel) measurable on $\sigma(T)$

$\rightsquigarrow$ $f(T)$ ($= U^* f U$ in Thm's notation)

makes sense in $W^*(T)$

Generally: take vectors $(v_i)_{i \in I}$ s.t.

$$\mathcal{H} = \bigoplus_{i \in I} \underbrace{C^*(T) v_i}_{\mathcal{H}_i}$$

$\rightsquigarrow$ By construction v_i is cyclic for $C^*(T)$

(or $C^*(T|_{\mathcal{H}_i})$; $\sigma(T|_{\mathcal{H}_i}) \subset \sigma(T)$)

$f(T|_{\mathcal{H}_i})$ for bdd Borel f makes sense

as op. on $\mathcal{H}_i$

$\rightsquigarrow$ put $f(T) = \bigoplus_{i \in I} f(T|_{\mathcal{H}_i}) \in \mathcal{L}(\mathcal{H})$.