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Summary
- Compact operators
- Diagonalization of selfadjoint cpt. ops.

Compact operators

($\mathcal{H}$: Hilbert sp.)

$\mathcal{F}(\mathcal{H}) = \{ T \in \mathcal{L}(\mathcal{H}) : \dim \operatorname{Ran} T < \infty \}$

$\mathcal{K}(\mathcal{H})$ = norm closure of $\mathcal{F}(\mathcal{H})$ in $\mathcal{L}(\mathcal{H})$.

Example : integral op. w/ cont. kernel.

$(T_k u)(x) = \int_0^1 k(x,y)\, u(y)\, dy$ for some cont. $k \in C([0,1]^2)$

$v_1, \dots, v_n, w_1, \dots, w_n \in C([0,1])$.

$k(x,y) = \sum_{i=1}^n v_i(x)\, \overline{w_i}(y)$

$\leadsto T_k u = \sum_{i=1}^n (u, w_i)_{L^2} \cdot v_i$

$\{ \sum_{i=1}^n v_i(x)\, w_i(y) : n = 1, 2, \dots, v_i, w_i \in C([0,1]) \}$

is dense in $C([0,1]^2)$ (Stone-Weier.)

$\| T_k - T_{k'} \|_{\mathcal{L}(L^2([0,1]))} \leq \| k - k' \|_\infty$

from $\int | \int k''(x,y)\, u(y)\, dy |^2 dx \leq \max |k''|^2 \, \|u\|_{L^2}^2$

$\Rightarrow T_k \in \mathcal{K}(\mathcal{H})$ for $k \in C([0,1]^2)$

Prop. $T \in \mathcal{K}(\mathcal{H}) \Leftrightarrow \forall X \subset \mathcal{H}$ (norm) bounded
TX has (norm) cpt closure

Proof. Step 0 We may assume $X = \mathcal{H}_1$

$\leadsto$ Enough to show : if $u_1, u_2, \dots$ are s.t. $\|u_n\| \leq 1$ then $(Tu_n)_n$ has a cluster point

Choose $T^{(1)}, T^{(2)}, \cdots$ in $\mathcal{F}(\mathcal{H})$ s.t.

· $\|T^{(n)}\| \le \|T\|$ · $n \le k, l \Rightarrow \|T^{(k)} - T^{(l)}\| < 2^{-n}$

$(\Leftarrow \|T - T^{(n)}\| < 2^{-(n+1)})$

Step 1 Find a conv. subseq. of $(T^{(1)} u_n)_n$

$\because$ Ran $T^{(1)}$ fin dim. $\Rightarrow T^{(1)} X$ has a cpt closure. $\Rightarrow \exists$ conv. subseq

$T^{(1)} u_{i_1}, T^{(1)} u_{i_2}, \cdots \to v_1$

Step 2 Find a conv. subseq. of $(T^{(2)} u_{i_n})_n$ from S.1.

$\because$ Ran $T^{(2)}$ is fin dim.

$\leadsto$ By cptness of $T^{(2)} X$ we can find subseq. $\to v_2$.

$$\|T^{(1)} u_{i_n} - T^{(2)} u_{i_n}\| \le \|T^{(1)} - T^{(2)}\| \cdot \|u_{i_n}\| \le 2^{-1}$$

$\Rightarrow \|v_1 - v_2\| \le 2^{-1}$

We continue to find subseq of $(T^{(k)} u_n)_n$ conv. to v_k, $\|v_k - v_{k+1}\| \le 2^{-k}$

Step "$\infty + 1$" $v_\infty = \lim v_k$ is a cluster pt of $(Tu_n)_n$. ☒

Cor : $T \in \mathcal{K}(\mathcal{H})$ $(\Leftarrow)$ any wkly conv. $(u_n)_n$ give norm conv. $(Tu_n)_n$

Pf. Step 1 $(u_n)_n$ is bounded.

$u_n^* \in \mathcal{H}^*$ by $u_n^*(v) = (v, u_n)$

We have $\forall v \in \mathcal{H}$ $(u_n^*(v))_n$ bdd

$\xRightarrow{\text{unif. bdd principle}} (\|u_n^*\|)_n$ bdd. but $\|u_n^*\| = \|u_n\|$

Step. 2 $\quad Tu_n \to Tu_\infty \quad$ for $\quad u_\infty = \text{wk-}\lim_n u_n$

$\because \quad (u_n)_n$ bdd (Step 1) $\underset{\text{Prop.}}{\Rightarrow}$ $(Tu_n)_n$ has a conv. subseq.

Let's say $Tu_{i_1}, Tu_{i_2}, \cdots \to w$

$$(v, w) = \lim_k (v, Tu_{i_k}) = \lim (T^*v, u_{i_k})$$

$$= (T^*v, u_\infty) = (v, Tu_\infty) \quad \text{for } \forall v$$

$\Rightarrow Tw = Tu_\infty \Rightarrow (Tu_n)_n$ can cluster only around Tu_∞. ☐

Th'm $\quad T$: cpt self adj op on $\mathcal{H}$.

$\exists$ orthonormal basis (u_n) $\quad Tu_n = \underset{\in \mathbb{R}}{\lambda_n} u_n$

$|\lambda_1| \geq |\lambda_2| \geq \cdots$

Proof. Step 1 $\quad \mathcal{H}_1 \to \mathbb{R}, \quad u \mapsto (Tu, u)$

takes max & min. (will show "max")

$\because \quad \mathcal{H}_1$ is weakly cpt $\quad (\mathcal{H} = \mathcal{H}^*)$

$\rightsquigarrow T\mathcal{H}_1$ is norm cpt. (Cor)

Choose $(u_n)_n \subset \mathcal{H}_1 \quad u_n \to u$ wk, $Tu_n \to Tu$ norm,

$$(Tu_n, u_n) \to \sup_{v \in \mathcal{H}_1} (Tv, v)$$

If we have $(Tu, u) = \lim_{n\to\infty} (Tu_n, u_n)$

u achieves the max.

$$|(Tu, u) - (Tu_n, u_n)| \qquad T = T^*$$

$$\leq |(T(u-u_n), u)| + |(Tu_n, u-u_n)|$$

$$\underset{\substack{\|u\|, \|u_n\| \leq 1 \\ \text{CSB}}}{\leq} \|T(u-u_n)\| \to 0$$

Step 2 $\quad \Lambda_{max} = \max_{u \in \mathcal{H}_1} (Tu, u)$, $\Lambda_{min} = \min_{u \in \mathcal{H}_1} (Tu, u)$

are either 0 or eig. val of T

$\because$ Suppose $\Lambda_{max} > 0$

Take $u \in \mathcal{H}_1$ $\quad \Lambda_{max} = (Tu, u)$

$\|u\| = 1$ by rescaling.

$Tu = \lambda u \iff \forall v \perp u \quad Tu \perp v$

We check the latter, we may assume $\|v\| = 1$

$u \perp v \Rightarrow \| \cos\theta\, u + \sin\theta\, v \| = 1$

put $f(\theta) = (T(\cos\theta\, u + \sin\theta\, v), \cos\theta\, u + \sin\theta\, v)$

f is max at $\theta = 0$. $\Rightarrow \partial_\theta f|_{\theta=0} = 0$

$\Rightarrow \underbrace{(Tv, u)}_{T = T^*} + (Tu, v) = 0.$

$\Rightarrow \mathrm{Re}(Tu, v) = 0.$ $\quad$ (for all $\|v\| = 1$, $u \perp v$)

$\Rightarrow (Tu, v) = 0$ for all $u \perp v$.

Step 3 $\quad$ We found a real eigenvec λ_1

Take u_1 s.t. $Tu_1 = \lambda_1 u_1$

T restr. to $\mathcal{H}^{(1)} = \{v \in \mathcal{H} : v \perp u_1\}$.

as cpt selfadj

$\rightsquigarrow$ has real eigenval $\lambda_2, \dots$ $\quad$ ☒