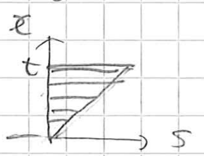
Application to diff. eq.

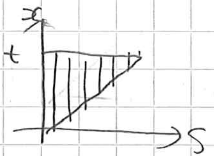
Fix $v \in L^2([0,1])$ consider the eq.

$$-\frac{d^2}{dt^2}u(t) = v(t), \qquad u(0) = 0 = u(1)$$

primitive func of v: $\int_0^x v(s)\,ds$

$$\rightsquigarrow u(t) = -\int_0^t \int_0^x v(s)\,ds\,dx = -\int_0^t \int_s^t v(s)\,dx\,ds$$





$$= \int_0^1 a(t,s)\,v(s)\,ds \qquad a(t,s) = \begin{cases} s-t & (t \geq s) \\ 0 & (t < s) \end{cases}$$

is one candidate. (but $u(1) \neq 0$)

$\rightarrow$ fix this u by linear func. to achieve $u(1) = 0$ so $u_1(t) = u(t) - u(1)t$.

$\rightarrow$ $u_1(t) = \int b(t,s)\,v(s)\,ds$

$b(t,s) = a(t,s) + t(1-s)$ is

1. cont. in s, t
2. integ. op: $(T_b v)(t) = \int b(t,s)\,v(s)\,ds$

gives sol. $-\frac{d^2 T_b v}{dt^2} = v$,

$(T_b v)(0) = (T_b(v))(1) = 0$

3. $b(s,t) = \overline{b(t,s)}$ $(= b(t,s))$

1. and 3. says T_b is a selfadj. cpt. op. $\Rightarrow$ real eigenvalues $(\lambda_i)_i$, $|\lambda_1| \geq \lambda_2 \geq \cdots$

$-\frac{d^2 u}{dt^2} = \mu u$, $u(0) = 0 = u(1)$ $\Rightarrow$ μ^{-1} is eigenv. of T

$u_k(t) = e^{2\pi\sqrt{-1}kt} \quad (k \in \mathbb{Z}) \quad$ ONB of $L^2([0,1])$

$$T_b u_k = \begin{cases} \frac{1}{(4k\pi)^2}(u_k - u_0) & k \neq 0 \\ \frac{1}{12} - \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(u_n + u_{-n}) & (k=0) \end{cases}$$

So in matrix form

$$\begin{bmatrix} & & & \vdots & & \\ & 0 & & -\frac{1}{4\pi^2} & 0 & \\ \cdots -\frac{1}{(4k\pi)^2} \cdots & & -\frac{1}{4\pi^2} & \frac{1}{12} & -\frac{1}{4\pi^2} & \cdots \\ & & & -\frac{1}{4\pi^2} & & \\ & 0 & & \vdots & 0 & \end{bmatrix}$$

Dirichlet problem

$$-\frac{d^2u}{dt^2}(t) + V(t)\,u(t) = \lambda\, u(t) \qquad V(t) \text{ real}$$

which λ admit solution for u?

i.e. (diff) op. $Q : C^2([0,1]) \to C([0,1])$

$$Qu = -\frac{d^2u}{dt^2} + V \cdot u$$

eigenvalue problem for

Prob. Q is not bdd on $L^2([0,1])$

$\rightsquigarrow$ look at Q^{-1}

Fix $\varepsilon > 0$ and define $S_\varepsilon \in \mathcal{K}(L^2([0,1]))$ by

$$S_\varepsilon u_k = \frac{1}{(2\pi k)^2 + \varepsilon} u_k \qquad u_k = e^{2\pi\sqrt{-1}kt}$$

cpt. pos. op.

(Why this? : $\varepsilon \to 0 \rightsquigarrow -\frac{d^2 S_\varepsilon u_k}{dt^2} = \frac{(2\pi k)^2}{(2\pi k)^2+\varepsilon} u_k \to u_k$

i.e. $S_\varepsilon \to \left(-\frac{\partial^2}{\partial t^2}\right)^{-1}$ (on compl. of scalars)

Put $F_\varepsilon = \sqrt{S_\varepsilon}$. $u_k \mapsto \frac{1}{\sqrt{(2\pi k)^2 + \varepsilon}} u_k$.

Lem. We may interpret $Q = F_\varepsilon^{-1}(1 + F_\varepsilon (V-\varepsilon) F_\varepsilon) F_\varepsilon^{-1}$

Pf. $Q u_k = (2\pi k)^2 u_k + V u_k$

RHS $= F_\varepsilon^{-2} + V - \varepsilon$, $F_\varepsilon^{-2} u_k = ((2\pi k)^2 + \varepsilon) u_k$

F_ε cpt (pos), $V - \varepsilon$ bdd real

$\Rightarrow$ $F_\varepsilon (V-\varepsilon) F_\varepsilon$ cpt self adj

$\Rightarrow$ has real eigenvals.

$V \geq \varepsilon \Rightarrow$ nonneg. eigenvals.

$\Rightarrow$ $1 + F_\varepsilon (V-\varepsilon) F_\varepsilon$ also has real / ≥ 1 eigenvalues.

Suppose this op. is invertible. ($\Leftarrow V \geq \varepsilon$)

Put $R = F_\varepsilon (1 + F_\varepsilon (V-\varepsilon) F_\varepsilon)^{-1} F_\varepsilon$

- "$RQ = 1 = QR$" by Lem.
- R is cpt, self adj
  - $\Rightarrow$ has real eigenvals $\to 0$.
  - $\exists$ ONB of eigenvectors.

$\Rightarrow$ Q has real eigenvals $\to \infty$

$\exists$ ONB of eigenvecs.

in particular $Qu = \lambda u$ only has sol. for ctbl λ.