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Summary
- Unbounded operators
  - closed operators
  - adjoint op.

Unbounded operator

$\mathcal{H}, \mathcal{H}'$ : Hilbert spaces

unbdd operator $T : \mathcal{H} \to \mathcal{H}'$ is given by
- domain $\mathcal{D}(T) \subset \mathcal{H}$. lin. subsp.
- linear map $\mathcal{D}(T) \to \mathcal{H}'$, $u \mapsto Tu$.

! important to keep track of $\mathcal{D}(T)$.

$T$ is densely defined if $\mathcal{D}(T) \subset \mathcal{H}$ dense.

Example differential op.
- $\mathcal{H} = \mathcal{H}' = L^2(\mathbb{R})$, $\mathcal{D}(T) = \{u \in C^1(\mathbb{R}) \cap L^2(\mathbb{R}) : u' \in L^2(\mathbb{R})\}$

  $(Tu)(x) = u'(x)$
- $\mathcal{H} = \mathcal{H}' = L^2([0,1])$ $\lambda \in \mathbb{C}$

  $\mathcal{D}(T_{(\lambda)}) = \{u \in C^1([0,1]) : u(1) = \lambda u(0)\}$

  $T_{(\lambda)} u = u'$.

$S, T$ ubdd ops. $T$ extends $S$ if

$\mathcal{D}(S) \subset \mathcal{D}(T)$, $T|_{\mathcal{D}(S)} = S$

we write $S \subset T$.

Basic operations $S, T$ ubdd
- $S + T$ domain $\mathcal{D}(S) \cap \mathcal{D}(T)$

  map $u \mapsto Su + Tu$.
- $ST$ domain $\{u \in \mathcal{D}(T) : Tu \in \mathcal{D}(S)\}$

  map $u \mapsto STu$.

Rem Even if $S, T$ are densely def'ed
$S+T$, $ST$ don't have to be so

Graph of $T$ : $\mathcal{G}_T = \{(u, Tu) : u \in \mathcal{H}\}$

$T$ is a <u>closed</u> operator if $\mathcal{G}_T$ is closed as a subset of $\mathcal{H} \oplus \mathcal{H}'$

This means : if $(u_n)_{n=1}^{\infty} \subset \mathcal{D}(T)$ is s.t.

$$\begin{cases} u_n \to u & (n \to \infty) \text{ in } \mathcal{H} \\ Tu_n \to v & \text{ in } \mathcal{H}' \end{cases}$$

then $u \in \mathcal{D}(T)$, $v = Tu$.

$T$ is <u>closable</u> if the closure of $\mathcal{G}_T$ is a graph of some map $\overline{T}$ from subset of $\mathcal{H}$

Lem. this is automatically linear

Pf. $u, v \in \mathcal{D}(\overline{T})$ $\lambda, \mu \in \mathbb{C}$

$\exists u_n \in \mathcal{D}(T)$ $u_n \to u$, $Tu_n \to \overline{T}u$.

similar for $v$

then $\lambda u_n + \mu v_n \in \mathcal{D}(T)$,

$\lambda u_n + \mu v_n \to \lambda u + \mu v$

$T(\lambda u_n + \mu v_n) \to \lambda \overline{T} u + \mu \overline{T} v$

So $(\lambda u + \mu v, \lambda \overline{T} u + \mu \overline{T} v) \in$ closure of $\mathcal{G}_T$.

$\Rightarrow \lambda u + \mu v \in \mathcal{D}(\overline{T})$, $\overline{T}(\lambda u + \mu v) = \lambda \overline{T} u + \mu \overline{T} v$.

$\overline{T}$ : closure of $T$ (closed by def)

$T$ closed $\leadsto$ $\mathcal{D}' \subset \mathcal{D}(T)$ is a <u>core</u> of $T$ if $T =$ closure of $T|_{\mathcal{D}'}$.

Inverse for <u>injective</u> $T$:

$\mathcal{D}(T^{-1}) = \operatorname{Ran} T$, $T^{-1}u = v$ if $Tv = u$.

Prop. $T$ inj, $\operatorname{Ran} T = \mathcal{H}'$, $T^{-1}$ bdd.

$\Rightarrow T$ closed.

Proof. $\mathcal{G}_T$ = "flip" of $\mathcal{G}_{T^{-1}}$.

$\because$ $S$ bdd $\mathcal{H}' \to \mathcal{H}$ $\Leftrightarrow$ $\mathcal{G}_S$ closed

(closed graph thm)

<u>Adjoint</u> of $T$ densely def'ed

$\mathcal{D}(T^*) = \{ u \in \mathcal{H}' : \mathcal{D}(T) \to \mathbb{C},\ v \mapsto (Tv, u)$ is bdd in norm top. $\}$

$T^*u$ vec. in $\mathcal{H}$

$(Tv, u) = (v, T^*u)$ for $v \in \mathcal{D}(T)$

Rem. $S \subset T \Rightarrow T^* \subset S^*$

$S^* + T^* \subset (S+T)^*$ $\quad T^*S^* \subset (ST)^*$.

( <u>if</u> these are densely def'ed)

$\operatorname{Ker} T^* = (\operatorname{Ran} T)^\perp$.

Prop 1) $T: \mathcal{H} \to \mathcal{H}'$ $\rightsquigarrow$ $T^*$ is closed

2) $T$ closable $\Leftrightarrow$ $T^*$ densely def'ed

and $\overline{T} = T^{**}$.

Proof Step 1 $U: \mathcal{H} \oplus \mathcal{H}' \to \mathcal{H}' \oplus \mathcal{H}$, $(u, v) \mapsto (-v, u)$

$\Rightarrow \mathcal{G}_{T^*} = U(\mathcal{G}_T^{\perp})$

$\because$ $(w, z) \in U \mathcal{G}_T^{\perp}$ means

$(w, -Tu)_{\mathcal{H}'} + (z, u)_{\mathcal{H}} = 0$

$\Leftrightarrow (w, Tu) = (z, u) \quad \forall u \in \mathcal{D}(T)$

This means $w \in \mathcal{D}(T^*)$, $z = T^* w$.

$\rightsquigarrow T^*$ is closed

Step 2 $\quad T$ closable $\Rightarrow \mathcal{D}(T^*)$ dense

$\because \quad \mathcal{G}_{\overline{T}} = \operatorname{clos} \mathcal{G}_T = \mathcal{G}_T^{\perp\perp} = (U^{-1}\mathcal{G}_{T^*})^{\perp}$

Enough to show $\mathcal{D}(T^*)^{\perp} = 0$ in $\mathcal{H}'$

$u \in \mathcal{D}(T^*)^{\perp} \Rightarrow (0, u) \in (U^{-1}\mathcal{G}_{T^*})^{\perp}$.

so $u = \overline{T}(0) = 0$.

Step 3 $\quad \mathcal{D}(T^*)$ dense $\Rightarrow T$ closable, $\overline{T} = T^{**}$

$\because \quad T^{**}$ is closed by Step 1

$\mathcal{G}_{T^{**}} = U^{-1} U \mathcal{G}_{T^{**}} \underset{\text{Step 1}}{=} U^{-1}(\mathcal{G}_{T^*}{}^{\perp}) = (U^{-1}\mathcal{G}_{T^*})^{\perp}$

$\underset{U^{-1}_{\mathcal{H},\mathcal{H}'} = -U_{\mathcal{H}',\mathcal{H}}}{=} (U\mathcal{G}_{T^*})^{\perp} = (\mathcal{G}_T)^{\perp\perp} = \overline{\mathcal{G}_T}$.

So $\overline{\mathcal{G}_T}$ is a graph of map, which is $T^{**}$.

$\boxtimes$

Ex. $\quad (T_{(\lambda)} u)(t) = u'(t)$ w/ $\mathcal{D}(T_{(\lambda)}) = \{u \in C^1([0,1]) \mid u(1) = \lambda u(0)\}$

$\mathcal{D}(T^*_{(\lambda)}) = \{u \in L^2[0,1] : u' \in L^2, \cancel{u(0) = 0 = u(1)}\}$

$T^*_{(\lambda)} u = -u'$ $\quad\quad u(t) = \int_0^t u'(s)\,ds = (u', \chi_{[0,t]})$

$u(1) = (\overline{\lambda})^{-1} u(0)$