Summary

- Symmetric operators, self adjoint operators
- ambiguity of selfadjoint extensions
  - deficiency index
  - ~~Friedrichs extension theorem~~

Symmetric ops

Densely defied op $T: \mathcal{H} \to \mathcal{H}$ is symmetric

if $(Tu, v) = (u, Tv)$ $\forall u, v \in \mathcal{D}(T)$

Self adjoint op. is $T: \mathcal{H} \to \mathcal{H}$ s.t.

$T = T^*$ (as unbounded ops.) so:

- $\mathcal{D}(T^*) = \{ u \in \mathcal{H} : \mathcal{D}(T) \to \mathbb{C}, v \mapsto (Tv, u)$ is bdd $\}$ is exactly eq. to $\mathcal{D}(T)$.
- $T$ is symmetric.

Rem $T$ symmetric $\Rightarrow$ closable

$\because$ we have $\mathcal{D}(T) \subset \mathcal{D}(T^*)$, and

$T^*$ $\Big|$ extends $T$ (from def.)
is closed (04.10)

$\rightsquigarrow$ $\underset{\text{closed}}{\mathcal{G}_T \subset \mathcal{G}_{T^*}}$ $\rightsquigarrow$ $\mathcal{G}_{\overline{T}} = \overline{\mathcal{G}_T} \subset \mathcal{G}_{T^*}$.

But $\overline{T}$ $(= T^{**}) \subsetneq T^*$.. can happen.

Essentially selfadj. op: symm. op $T$ s.t.

$\overline{T} = T^*$. (will be unique s.a. ext)

Ex. $T_{(\lambda)} u = u'(t)$ for

$\mathcal{D}(T_{(\lambda)}) = \{ u \in L^2([0,1]), u(t) = (u', \chi_{[0,t]}) + u(0)$ for some $u' \in L^2$, $u(1) = \lambda u(0) \}$

$\sqrt{-1}\, T_{(\lambda)}$ is self adj. if $\lambda \in \mathbb{T}$.

$\because$ Generally $\overline{T}_{(\lambda)}^* = -\overline{T}_{(\bar\lambda^{-1})}$ for $\lambda \neq 0$

$( \Leftrightarrow (\sqrt{-1}\,\overline{T}_{(\lambda)})^* = \sqrt{-1}\,\overline{T}_{(\bar\lambda^{-1})} )$

from $(\overline{T}_{(\lambda)}^* u, v) = (u, \overline{T}_{(\lambda)} v) = \int_0^1 u(t)\,\overline{v}'(t)\,dt$

$$= -\int_0^1 u'(t) v(t)\,dt + \underbrace{u(1)\,\overline{v}(1) - u(0)\,\overline{v}(0)}_{\text{would vanish if } u(1) = \bar\lambda^{-1} u(0)}$$

$\overline{v}(1) = \overline{\lambda \overline{v}(0)}$

$Su = u'$, $\mathcal{D}(S) = \{ u : \exists u' \in L^2, u(0) = 0 = u(1) \}$

$\sqrt{-1}S \subset \sqrt{-1}\,\overline{T}_{(\lambda)}$ for all $\lambda$, $\rightsquigarrow$ so no unique
symm.
self adj ext.

Deficiency index. (to capture ambiguity of self adj. ext.)

Lem $T$ closed symm., $\lambda \in \mathbb{C}$

then $\operatorname{Ker}(\lambda - T^*) = \operatorname{Ran}(\bar\lambda - T)^\perp$

Proof $\operatorname{Ker}(\lambda - T^*) = \{ u \in \mathcal{D}(T^*) : \lambda u = T^* u \}$

For such $u$ $(u, (\bar\lambda - T)v) = (\lambda u - T^* u, v) = 0$

for any $v \in \mathcal{D}(T)$ $\Rightarrow$ $u \in \operatorname{Ran}(\bar\lambda - T)^\perp$

If $u \in \operatorname{Ran}(\bar\lambda - T)^\perp$, for any $v \in \mathcal{D}(T)$

$(\lambda u, v) - (u, Tv) = (u, (\bar\lambda - T)v) = 0$

so $v \mapsto (u, Tv) = (\lambda u, v)$ is bdd

$\Rightarrow u \in \mathcal{D}(T^*)$, and we have $\lambda u = T^* u$

(by density of $\mathcal{D}(T)$)

Thm. $T$ closed symm. op. on $H$

- $\dim(\lambda - T^*)$ is const $(=: n_+(T))$ for $\operatorname{Im}\lambda > 0$
- $\sim$ $(= n_-(T))$ for $\operatorname{Im}\lambda < 0$

Proof Will do the case $\operatorname{Im}\lambda > 0$

Step 1. $\operatorname{Ran}(\bar\lambda - T)$ is closed

$\because$ $\bar\lambda - T = -\operatorname{Im}\lambda\sqrt{-1} - \underbrace{(T - \operatorname{Re}\lambda)}_{\text{symm}}$, same dom

& rescale by $(\operatorname{Im}\lambda)^{-1}$ $\rightsquigarrow$ WMA $\lambda = \sqrt{-1}$

$$\|(-\sqrt{-1} - T)u\|^2 = ((\sqrt{-1}+T)u, (\sqrt{-1}+T)u)$$

$$= (u,u) + \sqrt{-1}\underbrace{((u, Tu) - (Tu, u))}_{0} + (Tu, Tu)$$

$$\geq \|u\|^2 \quad \text{for } u \in \mathcal{D}(T)$$

If $v \in \overline{\operatorname{Ran}(-\sqrt{-1}-T)}$, any seq $(u_n)_n \subset \mathcal{D}(T)$ with $(-\sqrt{-1}-T)u_n \to v$ is a Cauchy seq.

$\Rightarrow$ $(\lim u_n, v) \in \overline{\mathcal{G}_{-\sqrt{-1}-T}} = \mathcal{G}_{-\sqrt{-1}-T}$.

$T$ clos. $\Rightarrow -\sqrt{-1}-T$ also

$\Rightarrow$ $v \in \operatorname{Ran}(-\sqrt{-1}-T)$

Step 2 if $|\mu|$ is small enough

$$\dim\operatorname{Ker}(\lambda+\mu-T^*) \leq \dim\operatorname{Ker}(\lambda-T^*)$$

Step 2-1. $\operatorname{Ker}(\lambda+\mu-T^*) \cap \operatorname{Ker}(\lambda-T^*)^\perp = 0$.

for small $\mu$.

$\because$ Suppose $u$ is in this, and $\|u\| = 1$

$$\operatorname{Ker}(\lambda-T^*)^\perp \underset{\text{Lem}}{=} \operatorname{Ran}(\bar\lambda - T)^{\perp\perp} \underset{\text{Step1}}{=} \operatorname{Ran}(\bar\lambda - T)$$

So $\exists\, v \in \mathcal{D}(T)$ : $u = (\bar\lambda - T)v$.

We also have $\mu u = (\lambda - T^*)u$, so

$$\underbrace{((\lambda - T^*)u, v)}_{\|u\|^2} = (\mu u, v)$$

$$|(u,v)| \underset{\text{CSB}}{\leq} \|u\|\cdot\|v\| \underset{\text{Step1}}{\leq} \|u\|\cdot|\operatorname{Im}\lambda|^{-1}\|u\|.$$

so $|\mu| < |\operatorname{Im}\lambda|$ fobids such $u$.

Step 2 : $P$ : orth. prj. to $\mathrm{Ker}(\lambda - T^*)$

$P|_{\mathrm{Ker}(\lambda+\mu - T^*)}$ is injective.

Step 3 $|\mu|$ small $\Rightarrow$ $\dim \mathrm{Ker}(\lambda+\mu-T^*)$
$\geq \dim \mathrm{Ker}(\lambda - T^*)$

$\because$ similar : enough to have
$|\mu| < \mathrm{Im}(\lambda+\mu)$ (like $|\mu| < \frac{1}{2}\mathrm{Im}\lambda$)

$\square$

So, $n_+(T) = \dim \mathrm{Ker}(\sqrt{-1} - T^*) = \dim \mathrm{Ran}(\sqrt{-1}+T)^\perp$
$n_-(T) = \dim \mathrm{Ker}(-\sqrt{-1} - T^*) = \dim \mathrm{Ran}(-\sqrt{-1}+T)^\perp$

Ex. "S" from prev. example $T = \sqrt{-1}S$
( $u \mapsto \sqrt{-1}u'$, $u(0)=0=u(1)$ )

$$\int_0^1 e^t(u(t)+u'(t))dt = \int_0^1 e^t u(t)dt + \int_0^1 e^t u'(t)dt$$

$$= e^t u(t)\Big|_0^1 - \int e^t u'(t)dt + \int e^t u'(t)dt$$

int. by parts

$\Rightarrow e^t \in \mathrm{Ran}(\sqrt{-1}S + \sqrt{-1})^\perp$
$e^{-t}$ ( —— $-\sqrt{-1})^\perp$

Of course $\mathrm{Ker}(\lambda - \underbrace{T^*}_{"\sqrt{-1}\frac{d}{dt}"}) = \langle e^{\sqrt{-1}\lambda t}\rangle$

Cor. $T$ closed symm.
$T$ is selfadj $\Longleftrightarrow$ $n_+(T) = 0 = n_-(T)$

Proof "$\Leftarrow$" enough to show $\mathcal{D}(T^*) \subset \mathcal{D}(T)$

$T+\sqrt{-1}$ is surj. by Lem. & $n_+(T)=0$
$u \in \mathcal{D}(T^*)$ $\Rightarrow$ $\exists\, v \in \mathcal{D}(T)$ s.t.
$(T^*+\sqrt{-1})u = (T+\sqrt{-1})v$ $\Rightarrow$ $u - v \in \mathrm{Ker}(T+\sqrt{-1})$
$T \subset T^*$
But $\dim \mathrm{Ker}(T+\sqrt{-1}) = n_-(T) = 0$ $\square$