Summary
- Understanding non-uniqueness of selfadjoint extensions
- Friedrichs ext. thm.

Non-uniqueness of s.a. extr.

$T : \mathcal{H} \to \mathcal{H}$ closed symm. op.

We want to understand
- selfadj. exts. ; if not unique find param. for all extr.
- in general; classify closed symm. exts $T \subset T'$

$\mathcal{K}_+ = \mathrm{Ker}(\sqrt{-1} - T^*)$, $\mathcal{K}_- = \mathrm{Ker}(-\sqrt{-1} - T^*)$

(so $n_\pm(T) = \dim \mathcal{K}_\pm$, T s.a. $\iff n_\pm(T) = 0$)

Lem. $\mathcal{D}(T^*) = \mathcal{D}(T) \oplus \mathcal{K}_+ \oplus \mathcal{K}_-$

right hand side is orth. for new inn. prd

$(u, v)' = (u, v) + (T^* u, T^* v)$

($\mathcal{D}(T^*) \hookrightarrow \mathcal{G}_{T^*}$, $u \mapsto u \oplus T^* u$)

Proof. Step 1. $\mathcal{D}(T) \perp \mathcal{K}_+$ for $(u, v)'$

$\because$ $u \in \mathcal{D}(T)$, $v \in \mathcal{K}_+$ ($\Rightarrow T^* v = \sqrt{-1} v$.)

$(u, v)' = (u, v) + (Tu, \sqrt{-1} v)$

$T \subset T^*$

$= (u, v) - \sqrt{-1}(u, T^* v) = (u, v) - (u, v) = 0$

Step 2 other orth. similar.

Step 3. $\mathcal{D}(T)$, $\mathcal{K}_\pm$ span $\mathcal{D}(T^*)$

$\because$ Take $u \in \mathcal{D}(T^*)$ orth. to $\mathcal{D}(T)$, $\mathcal{K}_\pm$.

(want $u = 0$)

$v \in \mathcal{D}(T)$ $(u, v)' = 0$ means

$(u, v) = -(T^* u, T v) = -(T^* T^* u, v)$

$\mathcal{D}(T) \subset \mathcal{H}$ dense $\Rightarrow$ $u = -T^* T^* u \Leftrightarrow ((T^*)^2 + 1) u = 0$

$(T^*)^2 + 1 = (T^* + \sqrt{-1})(T^* - \sqrt{-1}) \Rightarrow (T^* - \sqrt{-1}) u \in \mathcal{K}_-$

We have $(T^* - \sqrt{-1}) u = 0$

$\because$ $v \in \mathcal{K}_- \Rightarrow (v, (T^* - \sqrt{-1}) u)$

$= (\underbrace{T v}_{-\sqrt{-1} v}, u) + \sqrt{-1}\, \underline{(v, u)} = -\sqrt{-1} (v, u)' = 0$

$= -(T^* v, T^* u)$

$u \perp v$ for $(\ ,\ )'$

So $(T^* - \sqrt{-1}) u = 0 \Rightarrow u \in \mathcal{K}_+ \Rightarrow u = 0$. $\square$

$u \perp \mathcal{K}_+$

Thm. $\{$closed Symm. ext of $T\} \xleftrightarrow{1:1} \{$part. isom. $A : \mathcal{K}_+ \to \mathcal{K}_-\}$

$v \in \operatorname{Ran}(A^* A)$

$\mathcal{D}(T_A) = \{\underset{\mathcal{D}(T)}{u} + \underset{\mathcal{K}_+}{v} + \underset{\mathcal{K}_-}{A v}\} \longleftarrow A$

$T_A(u + v + Av) = Tu + \sqrt{-1} v - \sqrt{-1} A v.$

$n_\pm(T_A) = n_\pm(T) - \operatorname{rk}(A)$

Step 1 $T' \subset T^*$.

$\because$ $u \in \mathcal{D}(T') \Rightarrow \forall v \in \mathcal{D}(T)$ $(T'u, v) = (u, Tv)$

Symm, $T \subset T'$

so $v \mapsto (u, Tv)$ bdd on $\mathcal{D}(T)$.

i.e. $u \in \mathcal{D}(T^*)$

Step 2 Write $\mathcal{D}(T') = \mathcal{D}(T) \oplus \mathcal{K}'$ for $\mathcal{K}' \subset \mathcal{K}_+ \oplus \mathcal{K}_-$

$u \in \mathcal{K}'$ write $u = u_+ + u_-$ for $u_\pm \in \mathcal{K}_\pm$

then $\|u_+\| = \|u_-\|$.

Suit.

$\because$ We know $(T'u, u) = (u, T'u)$

, (T' symm.)

LHS : $\sqrt{-1}(u_+ - u_-, u)$

RHS $-\sqrt{-1}(u, u_+ - u_-)$

$\rightsquigarrow \sqrt{-1}\left((u_+, u_+) + (u_+, u_-) - (u_-, u_+) - (u_-, u_-)\right)$

$= -\sqrt{-1}\left((u_+, u_+) - (u_+, u_-) + (u_-, u_+) - (u_-, u_-)\right)$

$\Rightarrow (u_+, u_+) = (u_-, u_-)$

Step 4 $\mathcal{K}_0 = \{ u_+ \in \mathcal{K}_+ \quad \exists u_- \in \mathcal{K}_- \text{ s.t. } u_+ + u_- \in \mathcal{K}' \}$

def. A by $\mathcal{K}_0 \ni u_+ \mapsto u_-$ as above.

$\mathcal{K}_+ \ominus \mathcal{K}_0 \ni u_+ \mapsto 0$.

Well def part. iso by Step 3

Step 5. $T'(\underset{\mathcal{K}_+}{u + v} + \underset{\mathcal{K}_-}{Av}) = Tu + \sqrt{-1}v - \sqrt{-1}Av$, $T' \subset T^*$

Step 6 $n_\pm(T') = n_\pm(T) - \mathrm{rk}(A)$.

$\because$ $n_+(T) = \dim \mathrm{Ran}(\sqrt{-1} + T)^\perp$ (04.24)

$(\sqrt{-1} + T')(u + \underset{\mathrm{ran} A^*A}{v} + Av) = (\sqrt{-1} + T)u + 2\sqrt{-1}v$

So compl. is smaller by $\mathrm{ran} A^*A$

$\dim = \mathrm{rk} A$.

Cor. T closed symm. has a selfadj ext

iff $n_+(T) = n_-(T)$.

If this happens $T \subset T'$ sa $\overset{1:1}{\longleftrightarrow}$ unitary

$A : \mathcal{K}_+ \to \mathcal{K}_-$

Ex. $T_{(\lambda)} : L^2([0,1]) \to L^2([0,1])$, $u \mapsto u'$

$\mathcal{D}(T_{(\lambda)}) = \{ u \in L^2, u' \in L^2, u(1) = \lambda u(0) \}$

for $|\lambda| = 1$ extends T w/ $\mathcal{D}(T) = \{ \cdots \; u(0) = 0 = u(1) \}$

Friedrichs ext. th'm.

$T : \mathcal{H} \to \mathcal{H}$ is bdd from below by α if
(symm.) $(Tu, u) \geq \alpha \|u\|^2$ for $u \in \mathcal{D}(T)$.

(we write $T \geq \alpha$)

Th'm. $T : \mathcal{H} \to \mathcal{H}$ symm. $T \geq \alpha$ for some $\alpha \in \mathbb{R}$

$\Rightarrow$ $\exists$ s.a. $T' \supset T$ s.t. $T' \geq \alpha$.

Proof. Step 1 we may assume $\alpha = 1$

$\because$ rep. T by $T + (1-\alpha)$

Step 2. New inn. prod. $(u, v)' = (Tu, v)$
on $\mathcal{D}(T)$ is "bigger" than orig.
($\|u\|' \geq \|u\|$)

Step 3. $\mathcal{K}$: closure of $\mathcal{D}(T)$ for $\|\sim\|'$.

$T' = T^* |_{\mathcal{K} \cap \mathcal{D}(T^*)}$ ≥ 1.

$\because$ OK on dense subsp. $\mathcal{D}(T)$.

Step 4 T' is self adj.

$\because$ def $A : \mathcal{H} \to \mathcal{H}$ bdd by $A = JJ^*$
with $J : \mathcal{K} \to \mathcal{H}$ closure of incl.
($\|J\| \leq 1$ from $\|u\| \leq \|u\|'$)

J has dense img $\Rightarrow$ A is inj. bdd. pos.

$\Rightarrow$ A^{-1} is self adj. (closed by 04.16
$n_\pm(A^{-1}) = 0$ directly $\mathrm{Ran}(\pm\sqrt{-1} + A^{-1*})^\perp = 0$)

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