Summary
- Friedrichs ext. th'm.
- Laplacian.
- Cayley transform.
- Fredholm operators

Corr. from last time "Step 2" in Th'm

$(T'u, u) = (u, T'u)$ for $u \in K' = \mathcal{D}(T') \ominus \mathcal{D}(T)$

graph inn. pr.

$\because$ $T' \supset T$ is a symm. ext.

$\underset{\text{closed symm}}{T = T^{**}} \subset \underset{\substack{\text{sa = max} \\ \text{symm. ext}}}{T'} \subset \underset{\substack{\text{closed,} \\ \text{might not be symm.}}}{T^*}$

Friedrichs ext. (from last time's note)

⋮

Example. Laplacian.

$T = -\Delta = -\frac{d^2}{dt^2}$ on $L^2(\mathbb{R})$, $\mathcal{D}(T) = C_c^2(\mathbb{R})$

or $T = -\sum_{i=1}^n \frac{d^2}{dx_i^2}$ on $L^2(\mathbb{R}^n)$, $\mathcal{D}(T) = C_c^2(\mathbb{R}^n)$

$(Tu, u) \geq 0$ for $u \in \mathcal{D}(T)$

$\because$ $-\int_{-\infty}^{\infty} u''(t)\,\bar{u}(t)\,dt = \int_{-\infty}^{\infty} u'(t)\,\bar{u}'(t)\,dt = \|u'\|_{L^2}^2$

int. by parts

Cayley transform.

$T : \mathcal{H} \to \mathcal{H}$ self adj.

- $n_\pm(T) = \dim \operatorname{Ker}(\pm\sqrt{-1} - T) = \dim \operatorname{Ran}(\pm\sqrt{-1} + T)^\perp$ is zero.
- $\operatorname{Ran}(\pm\sqrt{-1} + T)$ is closed (4.24 Th'm Step 1)

$\Rightarrow$ $\kappa(T) = (T - \sqrt{-1})(T + \sqrt{-1})^{-1}$ is bij. lin $\mathcal{H} \to \mathcal{H}$.

Prop. $\kappa(T)$ is isometry (hence unitary)

Proof Any $u \in \mathcal{H}$ is written as $(T + \sqrt{-1})v$.

$\kappa(T)u = (T - \sqrt{-1})v$.

So it's enough to have $\|(T+\sqrt{-1})v\| = \|(T-\sqrt{-1})v\|$

$$\|(T+\sqrt{-1})v\|^2 = (Tv+\sqrt{-1}v, Tv+\sqrt{-1}v)$$

$$= (Tv, Tv) + \sqrt{-1}\underbrace{\left((v, Tv) - (Tv, v)\right)}_{0} + (v, v)$$

$$= \|(T-\sqrt{-1})v\|^2$$

∎

$f(t)$ (unbdd.) func. on $\mathbb{R}$.

$g(z) = f(\chi^{-1}(z))$ $\left(\chi^{-1}(z) = \sqrt{-1}\dfrac{1+z}{1-z}\right)$

func. on $\mathbb{T}$.

$f(T) = g(\chi(T))$ makes sense as a

closed op. on $\mathcal{H}$.

Ex. $U^s = \exp(\sqrt{-1}sT)$ for $s \in \mathbb{R}$

one par. group of unitaries $U^s U^{s'} = U^{s+s'}$

Fact Any strongly cont. one par. group

is like this $Tv = \dfrac{1}{\sqrt{-1}} \lim \dfrac{U^s v - v}{s}$

(Stone's theorem)

Fredholm operators

We want to capture "$\infty - \infty$ = finite":

how? connect "large portions" of bases by

a "nice" operator so they cancel in difference.

Ex. Unilateral shift $U : \ell_2\mathbb{N} \to \ell_2\mathbb{N}$, $(Ua)_n = a_{n-1}$
$(Ua)_0 = 0$

$(\,U : \delta_k \to \delta_{k+1}\,)$ $\dim \operatorname{Ker} U = 0$,
$\dim \operatorname{Cok} U = \dim \operatorname{Ran} U^{\perp} = 1$

U gives identification of δ_k in dom to

δ_{k+1} in codom.

$\dim \ell_2\mathbb{N} -_U \ell_2\mathbb{N}$ = $\dim$ "$-\mathbb{C}\delta_0$" $= -1$.
in codom

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A (bdd) op. $T : \mathcal{H} \to \mathcal{H}'$ is a **Fredholm** op.

if
- $\dim \operatorname{Ker} T < \infty$
- $\dim \operatorname{Cok} T := \dim \operatorname{Ker} T^* < \infty$
- $\operatorname{Ran} T \subset \mathcal{H}'$ is closed.

The **Fredholm** **index** of T is

$$\operatorname{Ind}(T) = \dim \operatorname{Ker} T - \dim \operatorname{Cok} T.$$