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Summary

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• Calkin algebra

$\mathcal{L}(\mathcal{H})$: alg of bounded ops. on $\mathcal{H}$

$\mathcal{K}(\mathcal{H})$: subalg of cpt. ops.

(closure of fin. rank ops.)

<u>Calkin algebra</u> $\mathcal{Q}(\mathcal{H}) = \mathcal{L}(\mathcal{H}) / \mathcal{K}(\mathcal{H})$

$= \{ [T] : T \in \mathcal{L}(\mathcal{H}) , \ [T] = [T'] \text{ iff } T - T' \in \mathcal{K}(\mathcal{H}) \}$

$[T] \cdot [T'] = [TT'] , \ [T] + [T'] = [T + T']$

Rem. This is a C^*-alg for $[T]^* = [T^*]$

$\| [T] \| = \min_{[T'] = [T]} \| T' \| = \min_{S \in \mathcal{K}(\mathcal{H})} \| T + S \|$

satisfies $\| [T]^* [T] \| = \| [T] \|^2$

Lem. $\mathcal{M}, \mathcal{N} \subset \mathcal{H}$ closed subsp. $\dim \mathcal{N} < \infty$

$\Rightarrow \mathcal{M} + \mathcal{N}$ is closed in $\mathcal{H}$

Proof What we use : unit ball $\mathcal{N}_1$ is cpt.

Step 1 we may assume $\mathcal{M} \cap \mathcal{N} = 0$

$\because$ Replace $\mathcal{N}$ by $\mathcal{N} \cap \mathcal{M}^\perp$

Step 2. If $u_n \in \mathcal{M}$, $v_n \in \mathcal{N}$ s.t.

$u_n + v_n \to w \ (n \to \infty)$, then $(v_n)_n$ bdd.

$\because$ Otherwise choose subseq. $v_{n_1}, v_{n_2}, \cdots \ \| v_{n_k} \| \to \infty$

By cptness of $\mathcal{N}_1$ $\frac{v_{n_k}}{\| v_{n_k} \|}$ has conv. subseq. $z = \lim_{j \to \infty} \frac{v_{m_j}}{\| v_{m_j} \|}$ $\| z \| = 1$.

$u_{m_j} + v_{m_j} \to w$ so $\frac{1}{\|v_{m_j}\|}(u_{m_j} + v_{m_j}) \to 0$

Then $\frac{u_{m_j}}{\|v_{m_j}\|} \to -z$. i.e. $z \in \mathcal{M} \cap \mathcal{N}$.

contradiction. (to $\mathcal{M} \cap \mathcal{N} = 0$)

Step 3 $w \in \mathcal{M} + \mathcal{N}$.

$\because$ We know $\|v_n\| \le R$ for some $R > 0$

By cptness of $\mathcal{N}_R$ $\exists$ subseq. $(v_{n_k})_k$

$v_{n_k} \to v_\infty \in \mathcal{N}_R$

$u_{n_k} + v_{n_k} \to w \Rightarrow u_{n_k} \to w - v_\infty$.

$\mathcal{M}$ closed $\Rightarrow w - v_\infty \in \mathcal{M}$

$\Rightarrow w = (w - v_\infty) + v_\infty \in \mathcal{M} + \mathcal{N}$. $\blacksquare$

Thm (Atkinson) $T : \mathcal{H} \to \mathcal{H}$ is Fredholm

$\Leftrightarrow [T] \in \mathcal{Q}(\mathcal{H})$ is invertible.

Proof $\Leftarrow$:

Step 1 $\exists$ $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{K}(\mathcal{H})$ s.t. $AT = I + B$.

$\because$ $[T]$ invertible $\Rightarrow \exists A$ s.t. $[A]\cdot[T] = [I]$

$\Rightarrow AT - I \in \mathcal{K}(\mathcal{H})$

Step 2 $\dim \operatorname{Ker} T < \infty$

$\because$ $Tu = 0 \underset{\text{Step 1}}{\Rightarrow} (I + B)u = 0 \Rightarrow u = -Bu \in \operatorname{Ran} B$.

$\rightsquigarrow$ $\underbrace{\operatorname{Ker} T}_{\text{Banach sp.}} = \operatorname{Ran}(\underbrace{P_{\operatorname{Ker} T}}_{\text{orth. prj}} B)$ cpt, open map by open map thm

$P_{\operatorname{Ker} T} B(\mathcal{H}_1)$ (contains an open ball / has norm cpt closure

open ball has norm cpt closure $\Leftrightarrow$ fin. dim

Step 3 $\quad \operatorname{Ker} T^* < \infty$

$\because \quad A' T^* = I + B'$ from invertibility of $[T]^*$

Step 4

Take C finite rank w/ $\| B - C \| < \frac{1}{2}$

$\Rightarrow \quad T \operatorname{Ker} C$ is closed

$\because \quad T|_{\operatorname{Ker} C}$ is bounded below

Suppose $u \in \operatorname{Ker} C$

$\| A \| \cdot \| T u \| \geq \| A T u \| = \| (I + B) u \|$

$= \| \underbrace{(I + C) u}_{u} + (B - C) u \| \geq \| u \| - \| (B - C) u \| > \frac{1}{2} \| u \|$

Step 5 $\quad \operatorname{Ran} T$ is closed

$\because \quad \operatorname{Ran} T = T(\operatorname{Ker} C) + T(\operatorname{Ker} C^{\perp})$

$\operatorname{Ker} C^{\perp}$ is fin. dim. (same as rank C)

$\rightsquigarrow$ Lem. is applicable.

$\Rightarrow$:

Step 1 $\quad T_0 = T|_{(\operatorname{Ker} T)^{\perp}} : (\operatorname{Ker} T)^{\perp} \to \operatorname{Ran} T$

is invertible.

$\because$ Bijective map between Banach sp.

$\Rightarrow$ inverse is bdd. (open mapping thm)

Step 2 $\quad$ Define $A \in \mathcal{L}(\mathcal{H})$ by

$A u = T_0^{-1} v \quad$ for $u = v + w$

$v \in \operatorname{Ran} T, \quad w \in (\operatorname{Ran} T)^{\perp}$

Then $[A] = [T]^{-1}$ in $\mathcal{Q}(\mathcal{H})$

$\because \quad AT = I - \underbrace{P_{\ker T}}, \quad TA = I - \underbrace{P_{(\mathrm{Ran}\, T)^\perp}}$

finite rank projs

Cor (of last step)

$$\mathrm{Ind}(T) = \mathrm{Tr}(\underbrace{I - TA}_{P_{\mathrm{cok}\, T}}) - \mathrm{Tr}(\underbrace{I - AT}_{P_{\ker T}})$$

Prop. Suppose $Q \in \mathcal{L}(\mathcal{H})$ also satisfies

$$QT - I, \quad TQ - I \quad \text{finite rank}$$

Then $\begin{cases} Q - A & \text{fin. rk} \\ \mathrm{Ind}\, T = \mathrm{Tr}(I - TQ) - \mathrm{Tr}(I - QT) \end{cases}$

Proof. Step 1 $Q - A$ fin. rk.

$$\because (Q - A) = \underbrace{(Q - A)T}_{\text{diff. of fin. rk} \Rightarrow \text{fin. rk.}} A + (Q - A)\underbrace{(I - TA)}_{\text{fin. rk.}}$$

Step 2 $\mathrm{Ind}\, T = \cdots$

$\because \quad \mathrm{Tr.}(TF - FT) = 0 \quad$ for $\quad \mathrm{rk}\, F < \infty$

Cor. rk F fin. $\quad \mathrm{Ind}(T + F) = \mathrm{Ind}(T)$.

$\because \quad (T + F)A - I, \quad A(T + F) - I \quad$ fin. rk.

Fact. We can repl. above by "$F \in \mathcal{K}(\mathcal{H})$"