Krein-Milman theorem

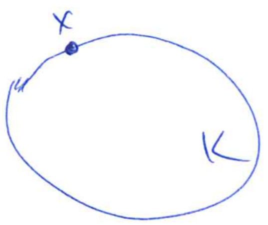
Let X be a vector space, $K \subset X$ a convex subset.

Def

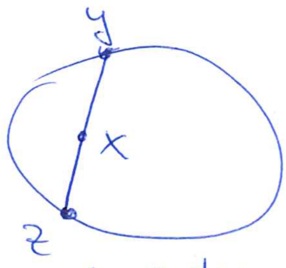
A point $x \in K$ is called extremal if whenever

$x = \lambda y + (1-\lambda) z$ for some $y, z \in K$, $0 < \lambda < 1$,

we must have $x = y = z$.



extremal



not extremal

Denote by $\operatorname{ex} K$ the set of extremal points of K.

For $\Omega \subset X$, we denote by $\operatorname{conv} \Omega$ the convex hull of Ω, that is, the set of elements of the form

$\lambda_1 x_1 + \ldots + \lambda_n x_n$, with $n \geq 1$, $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$,

$x_i \in \Omega$.

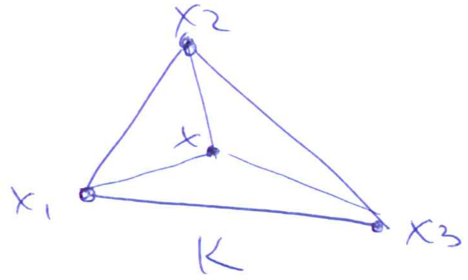
Theorem (Krein-Milman)

Assume X is a locally convex space, $K \subset X$ a compact convex subset. Then

$$K = \overline{\operatorname{conv}(\operatorname{ex} K)}.$$

Remark

For f.d. spaces the result was known long before Krein and Milman, and even in a stronger form: if $K \subset \mathbb{R}^n$ is compact and convex, then every point of K is a convex combination of at most $n+1$ extremal points.

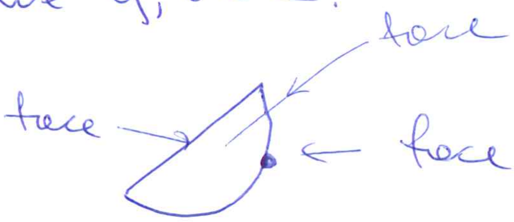


$\mathrm{ex}\, K = \{x_1, x_2, x_3\}$

For the proof of the theorem we will need the following notion.

<u>Def</u>

A subset $C \neq \emptyset$ of a convex set K is called a <u>face</u> if whenever $x = \lambda y + (1-\lambda) z$ for some $x \in C$, $y, z \in K$, $\lambda \in (0,1)$, we must have $y, z \in C$.



Thus $x \in K$ is extremal if and only if $\{x\}$ is a face.

<u>Proof of the Krein-Milman theorem</u>

<u>Claim</u> Every closed face C of K contains an extremal point of K.

To prove this, consider the collection $\mathcal{U}$ of all closed faces of K contained in C. Partially order $\mathcal{U}$ by the inverse inclusion. Every chain $\{C_i\}_i$ in $\mathcal{U}$ has an upper bound:

take $\bigcap_i C_i$, this set is nonempty by compactness of K, and it is easily seen to be a face.

By Zorn's lemma $\mathcal{U}$ has a maximal element C_0. Let us show that C_0 consists of one point, which is then necessarily an extremal point of K.

Take $f \in X^*$ and put

$$s = \inf_{C_0} \operatorname{Re} f \quad \text{and} \quad C_0' = \{x \in C_0 \mid \operatorname{Re} f(x) = s\}.$$

Note that as C_0 is compact, $C_0' \neq \emptyset$. The closed set C_0' is a face. Indeed, if $x = \lambda y + (1-\lambda) z$ for some $x \in C_0'$, $y, z \in K$, $0 < \lambda < 1$, then $y, z \in C_0$ as C_0 is a face. But then $\operatorname{Re} f(y) \geq s$, $\operatorname{Re} f(z) \geq s$ and $s = \operatorname{Re} f(x) = \lambda \operatorname{Re} f(y) + (1-\lambda) \operatorname{Re} f(z)$, hence $\operatorname{Re} f(y) = \operatorname{Re} f(z) = s$, so $y, z \in C_0'$.

As C_0 is a maximal element of $\mathcal{U}$, we conclude that $C_0' = C_0$. In other words, $\operatorname{Re} f$ is constant on C_0. Similarly $\operatorname{Im} f$ is constant, hence every $f \in X^*$ is constant on C_0.

As X^* separates points of the locally convex space X, it follows that C_0 consists of one point, proving our claim.

Applying the claim to $C = K$, we conclude that $\operatorname{ex} K \neq \emptyset$. Consider $K' = \overline{\operatorname{conv}(\operatorname{ex} K)}$. We have to show that $K' = K$.

Assume there exists $x_0 \in K \setminus K'$. Then by the Hahn-Banach separation theorem there exists $f \in X^*$ s.t.

$$\operatorname{Re} f(x_0) < \inf_{x \in K'} \operatorname{Re} f(x).$$

Put $s = \inf_{x \in K} \operatorname{Re} f(x)$ and $C = \{x \in K \mid \operatorname{Re} f(x) = s\}$.

Then, similarly to the argument above (for C_0'), C is a closed face in K, and $C \cap K' = \emptyset$.
By the Claim above, $C \cap \operatorname{ex} K \neq \emptyset$. But $\operatorname{ex} K \subset K'$.
This is a contradiction. Hence $K' = K$. ☒

Examples and applications

1) Let X be a locally compact space.
Consider $C_0(X)^*$. The (closed) unit ball in $C_0(X)^*$ is convex and w*-compact, hence it has many extreme points. Assume f is such a point.
Then $f(a) = \int_X a \, d\mu$ for a complex (regular, Borel) measure μ on X, and $|\mu|(X) = \|f\| \le 1$. We must have $\|f\| = 1$, as otherwise f is not extremal.
We claim that $\mu = z\delta_x$ for some $z \in \mathbb{T}$, $x \in X$.

First of all, we observe that for any Borel $A \subset X$ we have either $|\mu|(A) = 0$ or $|\mu|(X \setminus A) = 0$.
Indeed, otherwise, given A s.t. $0 < |\mu|(A) < |\mu|(X) = 1$, we could write

$$f = \lambda g + (1-\lambda) h, \text{ with } \lambda = |\mu|(A),\ g(a) = \frac{1}{|\mu|(A)} \int_A a \, d\mu,$$

$$h(a) = \frac{1}{|\mu|(X \setminus A)} \int_{X \setminus A} a \, d\mu.$$

Next, by regularity there is a compact $K \subset X$ s.t. $|\mu|(K) > 0$. But then $|\mu|(X \setminus K) = 0$ by our observation, hence $|\mu|(K) = 1$.

Next, let us show that there exists $x \in K$ s.t. $|\mu|(U) = 1$ for any neighbourhood U of x. If not, for every $x \in K$ we can find $U_x \ni x$ s.t. $|\mu|(U_x) = 0$. Choosing a finite subcover from $\{U_x\}_{x \in K}$ covering K, we conclude that $|\mu|(K) = 0$, which is a contradiction.

Now, if $x \in K$ is s.t. $|\mu|(U) = 1$ for all neighbourhoods $U \ni x$, by regularity we get $|\mu|(\{x\}) = 1$. Hence $|\mu| = \delta_x$. Then $\mu = z\delta_x$ for some $z \in \mathbb{T}$.

Conversely, we claim that $z\delta_x$ defines an extremal point of the unit ball of $C_0(X)^*$ for any $x \in X$, $z \in \mathbb{T}$. Indeed, assume

$$z\delta_x = \lambda\nu + (1-\lambda)\mu$$

for some complex measures ν, μ with $|\nu|(X) \le 1$, $|\mu|(X) \le 1$, and $0 < \lambda < 1$. Then

$$z = \lambda\nu(\{x\}) + (1-\lambda)\mu(\{x\})$$

and $|\nu(\{x\})| \le 1$, $|\mu(\{x\})| \le 1$. It follows that $\nu(\{x\}) = \mu(\{x\}) = z$. Together with $|\nu|(X), |\mu|(X) \le 1$, this is possible only when $\nu = \mu = z\delta_x$.

Thus we have proved that

$$\begin{matrix}\text{Extremal points of}\\ \text{the unit ball}\\ \text{of } C_0(X)^*\end{matrix} = \{z\delta_x \mid z \in \mathbb{T}, x \in X\}.$$

In particular, $\operatorname{conv}\{z\delta_x \mid z \in \mathbb{T}, x \in X\}$ is w^*-dense in the unit ball, which we already know.

2) Consider $\ell_1 = \ell_1(\mathbb{N})$.

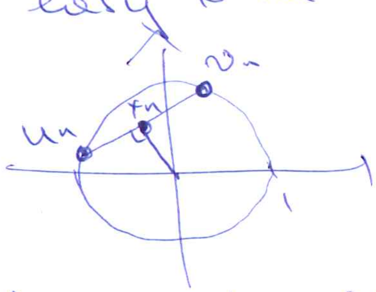
Then $\ell_1 = c_0^*$, so by the previous example applied to $X = \mathbb{N}$ we conclude that the extremal points of the unit ball of ℓ_1 are $z e_n$, $z \in \mathbb{T}$, $n \in \mathbb{N}$, where

$$e_n = (0, \dots, 0, \underset{n}{1}, 0, \dots).$$

On the other hand, consider $\ell^\infty = \ell_1^*$. Using that the extremal points of the unit disc in $\mathbb{C}$ are exactly the points on the unit circle, we easily conclude that the extremal points of the unit ball of ℓ^∞ is the set

$$\{ x = (x_n)_{n\geq 1}^\infty \in \ell^\infty \mid |x_n| = 1 \ \ \forall n \geq 1 \}.$$

Every element in the unit ball can be written as $\frac{1}{2}u + \frac{1}{2}v$ for extremal points u and v. This is easy to see coordinate-wise!



3) Consider a measure space $(\Omega, \mathcal{B}, \mu)$ (μ σ-finite). Similarly to the previous example, it is not difficult to see that the extremal points of the unit ball of $L^\infty(\Omega, \mathcal{B}, \mu)$ is the set

$$\{ f \in L^\infty \mid |f(x)| = 1 \text{ for } \mu\text{-a.e. } x \in \Omega \}.$$