What about $L^1(\Omega, \mathcal{B}, \mu)$?
Assume that μ is <u>nonatomic</u>, that is, $\forall A \in \mathcal{B}$ with $\mu(A) > 0$ $\exists B \in \mathcal{B}$ s.t.

$$B \subset A, \quad 0 < \mu(B) < \mu(A).$$

In this case the unit ball of $L^1(\Omega, \mathcal{B}, \mu)$ has no extremal points. Indeed, take $f \in L^1$, $\|f\|_1 \leq 1$. We may assume $\|f\|_1 = 1$, as otherwise f is clearly nonextremal. Consider

$$A = \{x \in \Omega \mid f(x) \neq 0\}.$$

Then $\int_A |f| \, d\mu = 1$, in particular, $\mu(A) > 0$. Choose $B \in \mathcal{B}$ s.t. $B \subset A$, $\mu(B) > 0$, $\mu(A \setminus B) > 0$. Then (compare with 1)) we can write

$$f = \lambda g + (1-\lambda) h,$$

where $g = \dfrac{f \chi_B}{\int_B |f| \, d\mu}$, $h = \dfrac{f \chi_{\Omega \setminus B}}{\int_{\Omega \setminus B} |f| \, d\mu}$, $\lambda = \int_B |f| \, d\mu$.

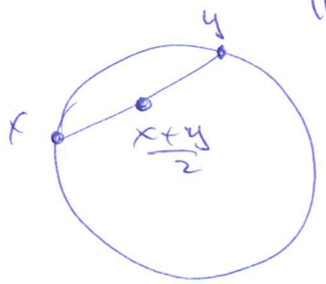
Thus f is not extremal.

In particular, it follows that $L^1(\Omega, \mathcal{B}, \mu)$ cannot be isometrically isomorphic to the dual of any Banach space.
(with a bit more effort "isometrically" can be omitted.)

4) Consider a measure space $(\Omega, \mathcal{B}, \mu)$ and $1 < p < \infty$. Then $L^p(\Omega, \mathcal{B}, \mu)$ is the dual of $L^q(\Omega, \mathcal{B}, \mu)$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, hence its unit ball has many extreme points.

Def

A normed vector space X is called uniformly convex if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, then
$$\left\| \tfrac{1}{2}x + \tfrac{1}{2}y \right\| \leq 1 - \delta.$$



Note that this implies that if $\|x\| = \|y\| = 1$, $x \neq y$, then $\|\lambda x + (1-\lambda)y\| < 1$ for $0 < \lambda < 1$. Indeed, if $\|\lambda x + (1-\lambda)y\| = 1$ for some $0 < \lambda < 1$, then, assuming that $0 < \lambda \leq 1/2$, we have
$$\lambda x + (1-\lambda)y = \tfrac{1}{2}y + \tfrac{1}{2}z, \text{ where } z = 2\lambda x + (1-2\lambda)y,$$
hence $\|z\| = 1$,
$z \neq y$, $\|z\| \leq 1$, $\|y\| = 1$, and this contradicts uniform convexity as we must have $\|\tfrac{1}{2}y + \tfrac{1}{2}z\| < 1$; the argument for $\tfrac{1}{2} < \lambda \leq 1$ is similar.

It follows that the extremal points of the unit ball in X are the points on the unit sphere.

Fact (Clarkson)
The spaces $L^p(\Omega, \mathcal{B}, \mu)$, $1 < p < \infty$, are uniformly convex.

In particular, the extremal points of the unit ball of L^p ~~are~~ is the set $\{ f \in L^p \mid \|f\|_p = 1 \}$.

Fact (Milman-Pettis)
Every uniformly convex Banach ~~normed~~ space is reflexive.

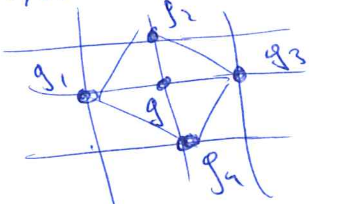
5) The Krein-Milman theorem can sometimes be used to understand a convex set by first understanding its extremal points.
As an example consider the following problem.
Let Γ be a discrete group and μ a finitely supported probability measure on Γ.

<u>Def</u>
A function $f : \Gamma \to \mathbb{C}$ is called <u>μ-harmonic</u> if

$$f(g) = \sum_h \mu(h) f(h^{-1}g) \quad \forall g \in \Gamma.$$

Is it possible to describe all harmonic functions?
For example, we can consider $\Gamma = \mathbb{Z}^2$,

$$\mu = \frac{1}{4}\left(\delta_{(0,1)} + \delta_{(1,0)} + \delta_{(0,-1)} + \delta_{(-1,0)}\right)$$



$\mathbb{Z}^2$

$$f(g) = \frac{1}{4}\left(f(g_1) + f(g_2) + f(g_3) + f(g_4)\right)$$

<u>Theorem</u> (Choquet-Deny)
Assume Γ is abelian and $\operatorname{supp}\mu$ generates the group Γ. Then every bounded μ-harmonic function $f : \Gamma \to \mathbb{C}$ is constant.

<u>Proof</u>
Consider the set $K \subset \ell^\infty(\Gamma)$ $(= \ell_1(\Gamma)^*)$ consisting of μ-harmonic functions f s.t. $\|f\|_\infty \le 1$.
It is easy to see that K is convex and w^*-closed. Since the unit ball of $\ell^\infty(\Gamma)$ is w^*-compact, it follows that K is w^*-compact, hence

$$K = \overline{\operatorname{conv}(\operatorname{ex} K)}^{w^*}$$

Assume $f \in \operatorname{ex} K$. Using that Γ is abelian we see that $f(h^{-1}\cdot)$ is μ-harmonic $\forall h \in \Gamma$, hence $f(h^{-1}\cdot) \in K$. But

$$f = \sum_{h \in \operatorname{supp}\mu} \mu(h) f(h^{-1}\cdot).$$

Since f is extremal, this implies that

$$f(h^{-1}\cdot) = f \quad \forall\, h \in \operatorname{supp}\mu.$$

But then for any $h_1, \dots, h_n \in \operatorname{supp}\mu$ we have

$$f(h_1^{\pm1} \dots h_n^{\pm1} \cdot) = f(h_2^{\pm1} \dots h_n^{\pm1} \cdot) = \dots = f.$$

It follows that $f(h\cdot) = f$ for all h in the subgroup generated by $\operatorname{supp}\mu$. But by assumption this subgroup is Γ. Hence $\underline{f(h) = f(e)}$ $\forall\, h \in \Gamma$, so f is constant. Then $K = \overline{\operatorname{conv}(\operatorname{ex} K)}^{w^*}$ also consists only of constant functions. $\boxtimes$

6) Here is another application.

<u>Def</u>

A matrix $A = (a_{ij})_{i,j=1}^n \in \operatorname{Mat}_n(\mathbb{R})$ is called <u>bistochastic</u> if $a_{ij} \geq 0$ $\forall\, i, j$ and

$$\sum_{i=1}^n a_{ij} = \sum_{i=1}^n a_{ji} = 1 \quad \forall\, j$$

<u>Theorem</u> (Birkhoff)

Every bistochastic matrix is a convex combination of permutation matrices.

Recall that a permutation matrix is a matrix of the form $a_{ij} = \begin{cases} 1, & i = \sigma(j) \\ 0, & \text{otherwise} \end{cases}$, where $\sigma \in S_n$.

<u>Proof</u>

The set of bistochastic n by n matrices is a compact convex subset of $\operatorname{Mat}_n(\mathbb{R}) = \mathbb{R}^{n^2}$. Hence it suffices to show that its extremal points are the permutation matrices.

Assume $A=(a_{ij})_{ij}$ is a bistochastic matrix and it is not a permutation matrix. Then $\exists\, i_1 j_1$ s.t.

$$0 < a_{i_1 j_1} < 1.$$

Since $\sum_i a_{i j_1} = 1$, it follows that there exists $i_2 \neq i_1$ s.t. $0 < a_{i_2 j_1} < 1$. We put $j_2 = j_1$.

Then, similarly, since $\sum_j a_{i_2 j} = 1$, $\exists\, j_3 \neq j_2$ s.t. $0 < a_{i_2 j_3} < 1$. We put $i_3 = i_2$. And so on.

We get a sequence

$(i_1, j_1), (i_2, j_2), \ldots$

Eventually we have to reach a smallest k s.t.

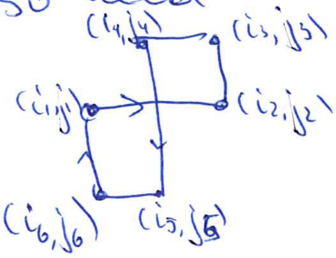
$(i_{k+1}, j_{k+1}) = (i_\ell, j_\ell)$ for some $\ell < k+1$.

By discarding $(i_1, j_1), \ldots, (i_{\ell-1}, j_{\ell-1})$, we may assume that $(i_{k+1}, j_{k+1}) = (i_1, j_1)$.

Next, if $(i_k, j_k), (i_1, j_1), (i_2, j_2)$ happen to be in the same row or column, we also discard (i_1, j_1).

Thus we get $(i_1, j_1), \ldots, (i_k, j_k), (i_{k+1}, j_{k+1}) = (i_1, j_1)$

s.t. with every move we change row and keep column, then change column and keep row, and so on. The number k must be even, since our path turn by 90° with every move, so need an even number of turns to return back.



Choose $\varepsilon > 0$ s.t. $\varepsilon < a_{i_\ell, j_\ell}$, $\varepsilon < 1 - a_{i_\ell, j_\ell}$ for all $\ell = 1, \dots, k$. Consider new matrices B, C defined by

$$b_{ij} = \begin{cases} a_{ij} + \varepsilon, & \text{if } (i,j) = (i_\ell, j_\ell) \text{ for an odd } \ell, \\ a_{ij} - \varepsilon, & \text{if } (i,j) = (i_\ell, j_\ell) \text{ for an even } \ell, \\ a_{ij}, & \text{otherwise.} \end{cases}$$

$$c_{ij} = \begin{cases} a_{ij} - \varepsilon, & \text{if } (i,j) = (i_\ell, j_\ell) \text{ for an even } \ell \\ a_{ij} + \varepsilon, & \text{if } (i,j) = (i_\ell, j_\ell) \text{ for an odd } \ell \\ a_{ij}, & \text{otherwise} \end{cases}$$

Then B and C are bistochastic and $A = \frac{1}{2}(B + C)$, so A is not extremal. $\boxtimes$

Thus every extremal bistochastic matrix must be a permutation matrix. It is also easy to see that every permutation matrix is indeed extremal. $\boxtimes$