MAT4450: EXERCISE PROBLEMS

'Exercise' numbers refer to those of Douglas's book.

Problem 1 (Exercise 1.25)**.** Let $\mathscr{X}$ be a Banach space. Show that $\mathscr{X}$ is a dense subspace of $\mathscr{X}^{**}$ with respect to the weak*-topology, i.e., the $\sigma(\mathscr{X}^{**}, \mathscr{X}^*)$-topology.

Hint. For example, try to showing that the w*-closure $\mathscr{Y}$ of $\mathscr{X}$ is equal to $\mathscr{X}^{**}$ by contradiction: if $\mathscr{X}^{**}/\mathscr{Y}$ is nontrivial, a functional on this space corresponds to a w*-continuous functional on $\mathscr{X}^{**}$ which vanishes on $\mathscr{X}$. What are such functionals?

Problem 2 (Exercise 1.33)**.** Let $\mathscr{X}$ be a Banach space. Show that a linear form $\omega\colon \mathscr{X}^* \to \mathbb{C}$ is continuous for the weak*-topology if and only if its restriction to the unit ball $\mathscr{X}^*_1$ is continuous for the (restriction of) weak*-topology.

Hint. We want to show that, if ω is w*-continuous on $\mathscr{X}^*_1$, then it is given by an evaluation at some $v \in \mathscr{X}$. Explain that the Hahn–Banach theorem implies that there is an (isometric) embedding $\mathscr{X} \to C(\mathscr{X}^*_1)$.

In the following $\mathscr{X} \otimes \mathscr{Y}$ denotes the algebraic tensor product of $\mathscr{X}$ and $\mathscr{Y}$.

Problem 3 (Exercises 1.37 and 1.39)**.** For $z \in \mathscr{X} \otimes \mathscr{Y}$, its *injective* norm (or inductive norm, ϵ-norm) is defined by

$$\|z\|_\epsilon = \sup\left\{ \sum_{i=1}^n |\phi(x_i)\psi(y_i)| \;\middle|\; \phi \in \mathscr{X}^*_1, \psi \in \mathscr{Y}^*_1 \right\}$$

where $z = \sum_{i=1}^n x_i \otimes y_i$ (this definition does not depend on the choice of this expression). When X and Y are compact topological spaces, show that the inclusion

$$C(X) \otimes C(Y) \to C(X \times Y), \quad f \otimes g \mapsto h(x,y) = f(x)g(y)$$

induces the injective norm.

Hint. The supremum will be attained by extremal points of $C(X)^*_1$ and $C(Y)^*_1$. What are those?

Problem 4 (Exercises 1.36 and 1.40)**.** For $z \in \mathscr{X} \otimes \mathscr{Y}$, its *projective* norm (or π-norm) is defined by

$$\|z\|_\pi = \inf\left\{ \sum_{i=1}^n \|x_i\| \, \|y_i\| \;\middle|\; z = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

(This time $\sum_{i=1}^n \|x_i\| \, \|y_i\|$ will depend on representatives $(x_i)_i$ and $(y_i)_i$.) When X and Y are compact topological spaces, show that the above inclusion $C(X) \otimes C(Y) \to C(X \times Y)$ induces a homeomorphism from the completion for projective norm (if and) only if at least one of X and Y is finite.

Hint. If X and Y both have N points, one can choose $(f_i)_{i=1}^N$ and $(g_i)_{i=1}^N$ (supported on small neighborhoods of those points) such that

$$\Big\| \sum_i f_i \otimes g_i \Big\|_\pi \geq N, \qquad \Big\| \sum_i f_i \otimes g_i \Big\|_{C(X \times Y)} = 1.$$

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