MAT4450: EXERCISE PROBLEMS

'Exercise' numbers refer to those of Douglas's book.

Problem 1 (Exercise 3.2)**.** Show that the parallelogram law for norm implies existence of Hermitian inner product.

Problem 2 (Exercise 3.4)**.** Show that $C([0,1])$ is not isomorphic to a Hilbert space (as a Banach space).

Hint. Check that the uniform norm does not satisfy the parallelogram law, by looking at functions with nonintersecting support.

Problem 3 (Exercise 3.9)**.** Let $\mathscr{H}$ be a Hilbert space, and $\mathscr{K}$ be its subspace. If ϕ is a bounded functional on $\mathscr{K}$, show that it has a unique extension ϕ' to $\mathscr{H}$ with $\|\phi'\| = \|\phi\|$.

Hint. Extend ϕ to the closure $\mathscr{K}'$ of $\mathscr{K}$ first. We have the decomposition $\mathscr{H} = \mathscr{K}' \oplus \mathscr{K}^\perp$, but how should we define ϕ' on $\mathscr{K}^\perp$ to achieve $\|\phi'\| = \|\phi\|$?

Problem 4 (Exercise 3.11)**.** Let $\mathscr{H}$ be a Hilbert space. Show that the unit vectors are the extreme points of the unit ball of $\mathscr{H}$.

Hint. Show that $\|u+v\| < \|u\| + \|v\|$ unless u and v are positive scalar multiple of each other.

Problem 5 (Exercise 2.6)**.** Let $\mathscr{X}$ be a Banach space, and $\mathcal{L}(\mathscr{X})$ be the ring of bounded linear transforms on $\mathscr{X}$. Show that the space $\mathcal{F}(\mathscr{X})$ of *finite rank* linear transforms on $\mathscr{X}$ is a two-sided ideal of $\mathcal{L}(\mathscr{X})$.

Hint. The essential point is to show that, if $T \in \mathcal{L}(\mathscr{X})$ and $S \in \mathcal{F}(\mathscr{X})$, the transforms ST and TS are in $\mathcal{F}(\mathscr{X})$.
Extra problem: let $\mathcal{K}(\mathscr{X})$ be the norm closure of $\mathcal{F}(\mathscr{X})$ inside $\mathcal{L}(\mathscr{X})$ (the space of *compact* linear transforms). Show that $\mathcal{K}(\mathscr{X})$ is also a two-sided ideal of $\mathcal{L}(\mathscr{X})$.

If X is a *locally compact* topological space, one can consider the commutative Banach algebra

$$C_b(X) = \{f \colon X \to \mathbb{C} \mid \text{continuous and bounded}\},$$

but (usually) it is more sensible to consider its subspace

$$C_0(X) = \{f \colon X \to \mathbb{C} \mid \text{continuous}, \forall \epsilon > 0 \exists K \subset X \text{ compact } \forall x \notin K \colon |f(x)| < \epsilon\},$$

which is again a commutative Banach algebra, without unit if X is noncompact. (One can write the above condition as $\lim_{x\to\infty} f(x) \to 0$.)

Problem 6. Consider the case of $X = (0,1)$ (open unit interval). Let $\mathscr{A}$ be the linear span of $C_0((0,1))$ and $\mathbb{C}$ inside $C_b((0,1))$ can be identified with $C(\mathbb{T})$ as a Banach algebra.

Hint. We want to identify $0 < t < 1$ with $e^{2\pi\sqrt{-1}t} \in \mathbb{T}$. Write down the induced linear map $C_0((0,1)) \to C(\mathbb{T})$, and check that it extends to an Banach algebra isomorphism $\mathscr{A} \to C(\mathbb{T})$.

Problem 7 (Exercise 2.11)**.** Let X be a locally compact space, and take the set $M = M_{C_b(X)}$ of multiplicative functionals on $C_b(X)$, endowed with the weak*-topology. Show that the Gelfand transform $C_b(X) \to C(M)$ is *isometric*.

Hint. To show that it's contractive, use $\phi \in M \Rightarrow \|\phi\| \leq 1$. To show that it does not decrease the norm, give an map $X \to M$.
The above problem shows that $C_b(X)$ is isomorphic to $C(M)$. We write $\beta X = M$, and call it the *Čech compactification* of X.

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