MAT4450: EXERCISE PROBLEMS

References:

- B. C. Hall, *Quantum theory for mathematicians*, Grad. Text in Math. **267**, Springer, New York, 2013.
- G. K. Pedersen, *Analysis now*, Grad. Text in Math. **118**, Springer-Verlag, New York, 1989.

Problem 1. Consider the unbounded operator $T\colon L^2([0,1]) \to L^2([0,1])$ by

$$\mathcal{D}(T) = \left\{ f(t)\colon \text{restriction of periodic function } f(t) = f(t+1) \text{ in } C^1(\mathbb{R}) \right\}, \quad Tf = f'.$$

Using the Fourier transform, describe a corresponding operator $S\colon \ell_2\mathbb{Z} \to \ell_2\mathbb{Z}$. (You can ignore the precise identification of $\mathcal{D}(S)$.) Describe the closure of S, and explain how it translates back on $L^2([0,1])$.

Hint. $S+1$ will be diagonalizable with nonzero eigenvalues, and will have bounded inverse.

Problem 2 (P. 5.1.7 and 5.1.8)**.** Let T be an injective bounded selfadjoint operator on $\mathscr{H}$. Then T^{-1} is an (possively unbounded) selfadjoint operator.

Problem 3 (P. 5.1.14)**.** Consider the unbounded operator $T\colon L^2(\mathbb{C}) \to L^2(\mathbb{C})$ by

$$\mathcal{D}(T) = \left\{ f(z)\colon \left\| z f(z) \right\|_{L^2} < \infty \right\}, \quad (Tf)(z) = z f(z).$$

What is T^*? (Don't skip the identification of $\mathcal{D}(T^*)$ for this.)

Problem 4. Consider two operators $S, T\colon \ell_2\mathbb{N} \to \ell_2\mathbb{N}$ given as $\mathcal{D}(S) = \mathcal{D}(T) = \left\{ a \in \ell_2\mathbb{N}\colon \sum_n n^4 \left| a_n \right|^2 < \infty \right\}$,

$$(Sa)_n = n^2 a_n \quad (n > 1), \qquad (Sa)_1 = \sum_{k=1}^{\infty} k a_k, \qquad (Ta)_n = -n^2 a_n \quad (n > 1), \qquad (Ta)_1 = 0.$$

Show that these are closed operators, but $S+T$ is not closable.

Hint. Look at the sequence $u_n = \frac{1}{n}\delta_n$ for $n = 1, 2, \dots$ in $\ell_2\mathbb{N}$ for the second statement.

For the next two problems, the ambient Hilbert space is $\mathscr{H} = L^2(\mathbb{R})$, and $C_c^\infty(\mathbb{R})$ is the space of compactly supported smooth functions on $\mathbb{R}$.

Problem 5 (H. Proposition 9.29)**.** Consider a symmetric operator $T\colon \mathscr{H} \to \mathscr{H}$ defined by

$$\mathcal{D}(T) = C_c^\infty(\mathbb{R}), \quad (Tu)(x) = \sqrt{-1}u'(x).$$

Problem 6 (H. Section 9.10)**.** Consider a symmetric operator $T\colon \mathscr{H} \to \mathscr{H}$ defined by

$$\mathcal{D}(T) = C_c^\infty(\mathbb{R}), \quad (Tu)(x) = -u''(x) - x^4 u(x).$$

Show that T is not essentially selfadjoint by following these steps:

(1) for $\alpha > 0$, put $p_\alpha(x) = \sqrt{x^4 + \sqrt{-1}\alpha}$ (take a suitable branch). Then the function

$$v_\alpha(x) = \frac{1}{\sqrt{p_\alpha(x)}} \exp\left(\sqrt{-1} \int_0^x p_\alpha(y) dy \right)$$

and also 'Tv_α' $= -v_\alpha''(x) - x^4 v_\alpha(x)$ (interpreted in an obvious way), belong to $L^2(\mathbb{R})$.

(2) v_α belongs to $\mathcal{D}(T^*)$, and satisfies

$$\left\| (T^* - \sqrt{-1}\alpha) v_\alpha \right\|^2 < \alpha^2 \left\| v_\alpha \right\|^2$$

if α is sufficiently large. (Try to bound $|Tv_\alpha|$ by $C\,|v_\alpha|$.)

(3) if T was essentially selfadjoint, T^* would be its closure and in particular closed symmetric. After rescaling by α^{-1}, the above estimate violates 'Step 1' of Theorem from April 24 lecture.

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