MAT4450 - Spring 2024 - Exercises - Set 1

We first recall some terminology. Let X be a topological space. A subset V of X is called a *neighborhood of* $x \in X$ whenever V contains an open set to which x belongs. We denote the family of all neighborhoods of x by $\mathcal{N}_x$ (and call it the *neighborhood system of* x).[1] By *neighborhood basis at* x we mean a collection $\mathcal{B}_x$ of neighborhoods of x (i.e., $\mathcal{B}_x \subseteq \mathcal{N}_x$) such that each neighborhood of x contains at least one element of $\mathcal{B}_x$.

Exercise 1. Assume X be a topological space which is *first countable*, that is, every $x \in X$ has a countable neighborhood basis. Show that the topology of X may then be described with the help of sequences, by showing that for each $A \subseteq X$ and $x \in X$ we have

$$x \in \overline{A} \Leftrightarrow \text{there exists a sequence } \{x_n\} \text{ in } A \text{ such that } x_n \to x \text{ as } n \to \infty.$$

Exercise 2. Let $X = \mathcal{F}(\mathbb{R}, \mathbb{R})$ consists of all real-valued functions on $\mathbb{R}$ and equip it with the topology of pointwise convergence. We recall that a neighborhood basis $\mathcal{B}_f$ at $f \in X$ is given by

$$\mathcal{B}_f = \big\{V_{f,S,\varepsilon} : S \text{ is a finite subset of } \mathbb{R},\ \varepsilon > 0\big\},$$

where $V_{f,S,\varepsilon} := \{g \in X : |f(s) - g(s)| < \varepsilon \text{ for every } s \in S\}$.

For $h \in X$ we set $\operatorname{supp}(h) := \{t \in \mathbb{R} : h(t) \neq 0\}$. Moreover, we set

$$A := \{h \in X : \operatorname{supp}(h) \text{ is finite}\}.$$

a) Show that $\overline{A} = X$ (i.e., A is dense in X) in the following two ways:

i) by showing that for each $f \in X$ we have $V \cap A \neq \emptyset$ for every $V \in \mathcal{N}_f$;

ii) by showing that for each $f \in X$ there exists a net in A which converges to f.

b) Choose any $f \in X$ such that $\operatorname{supp}(f)$ is uncountable (e.g., $f(t) = t$ for all $t \in \mathbb{R}$). Note that $f \in \overline{A}$ by a). Show that no sequence in A converges to f.

Exercise 3. Let X be a nonempty set and $A \subseteq X$. Consider a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in X. Check that the following statements are true:

a) If $\{x_\alpha\}$ is eventually in A, then it is frequently in A.

b) $\{x_\alpha\}$ is not frequently in A if and only if it is eventually in $X \setminus A$.

c) If $\{x_\alpha\}$ is eventually in A, then it is not eventually in $X \setminus A$, and if $\{x_\alpha\}$ is eventually in $X \setminus A$, then it is not eventually in A.

Exercise 4. Let X be a topological space and let $\{x_\alpha\}_{\alpha \in \Lambda}$ be a universal net in X (so it is eventually in A or eventually in $X \setminus A$ for every $A \subseteq X$).

Assume that $x \in X$ is a cluster point[2] of $\{x_\alpha\}$, that is, $\{x_\alpha\}$ is frequently in V for every $V \in \mathcal{N}_x$. Check that $\{x_\alpha\}$ converges to x.

(This implies that if a universal net in X has some cluster points, then it converges to every of its cluster points.)

If you have time, you should also have a look at Exercise 1.3.3 in Pedersen's book, to realize how Riemann-integrability can be formulated in terms of net-convergence.

[1] Some authors denote the family $\mathcal{N}_x$ by $\mathcal{O}(x)$ and call it the neighborhood filter of x. Note also that it is sometimes required that a neighborhood of a point is open, but we don't; this is essentially a matter of taste.

[2] Cluster points are called accumulation points in some books.