MAT4450 - Spring 2024 - Exercises - Set 10

Exercise 43

Solve Exercise 4.1.13 in Pedersen's book.

Exercise 44

Consider the unital commutative Banach algebra $\mathcal{A} = \ell^1(\mathbb{Z}, \mathbb{C})$ (w.r.t. the $\|\cdot\|_1$-norm). Set $\mathcal{B} := \{f \in \mathcal{A} : f(n) = 0 \text{ for all } n < 0\}$.

a) Check that $\mathcal{B}$ is a norm-closed subalgebra of $\mathcal{A}$ which contains the unit of $\mathcal{A}$. (Hence $\mathcal{B}$ is also a unital commutative Banach algebra.)

b) Let $\lambda \in \mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$. Check that the map $\gamma_\lambda : \mathcal{B} \to \mathbb{C}$ given by

$$\gamma_\lambda(f) = \sum_{n=0}^{\infty} f(n)\lambda^n$$

for all $f \in \mathcal{B}$ is well-defined and belongs to $\widehat{\mathcal{B}}$.

c) Show that the map $\lambda \mapsto \gamma_\lambda$ gives a homeomorphism from $\mathbb{D}$ onto $\widehat{\mathcal{B}}$. Then, identifying $\widehat{\mathcal{B}}$ with $\mathbb{D}$, check that the Gelfand transform of $\mathcal{B}$ is the map $\Gamma : \mathcal{B} \to C(\mathbb{D})$ given by

$$[\Gamma(f)](\lambda) = \sum_{n=0}^{\infty} f(n)\lambda^n \quad \text{for } f \in \mathcal{B} \text{ and } \lambda \in \mathbb{D}.$$

Deduce that Γ is one-to-one and describe the range of Γ.