MAT4450 - Spring 2024 - Exercises - Set 11

Exercise 45

Assume Ω is a non-compact, locally compact Hausdorff space. Consider the set of complex continuous functions on Ω given by $\mathcal{A} = C_0(\Omega) + \mathbb{C}\,1_\Omega$.

a) Check that $\mathcal{A}$ is a unital commutative C^*-algebra w.r.t. $\|\cdot\|_\infty$.

b) Show that $\widehat{\mathcal{A}}$ may be identified with the one-point compactification $\Omega \cup \{\infty\}$ of Ω, and deduce that $\mathcal{A}$ is isometrically $*$-isomorphic to $C(\Omega \cup \{\infty\})$.

Exercise 46

Let Ω, Ω' be topological spaces. Recall that $C_b(\Omega)$ denotes the commutative unital C*-algebra consisting of all complex bounded continuous functions on Ω, equipped with the $\|\cdot\|_\infty$-norm. We denote the character space $\widehat{C_b(\Omega)}$ of $C_b(\Omega)$ by $\beta\Omega$. By Gelfand's theorem, $\beta\Omega$ is a compact Hausdorff space.

a) For $\omega \in \Omega$, define $\iota_\omega : C_b(\Omega) \to \mathbb{C}$ by $\iota_\omega(f) = f(\omega)$ for all $f \in C_b(\Omega)$. Check that $\iota_\omega \in \beta\Omega$. Then check that the map $\omega \mapsto \iota_\omega$ from Ω into $\beta\Omega$ is continuous.

b) Let $h : \Omega \to \Omega'$ be a continuous map. Define a map $\Phi_h : C_b(\Omega') \to C_b(\Omega)$ by

$$\Phi_h(g) = g \circ h$$

for each $g \in C_b(\Omega')$.

i) Check that Φ_h is a bounded $*$-homomorphism satisfying $\|\Phi_h\| = 1$.

ii) Check that Φ_h is isometric (and therefore injective) when h is surjective.

iii) Assume that Ω, Ω' are both compact Hausdorff spaces. Use Tietze's extension theorem (cf. Munkres' or Pedersen's book) to show that Φ_h is surjective whenever h is injective. Deduce that Φ_h is an isometric $*$-isomorphism from $C(\Omega')$ onto $C(\Omega)$ whenever h is a homeomorphism.

c) Assume $\Phi : C_b(\Omega') \to C_b(\Omega)$ is an (algebra-)homomorphism such that $\Phi(1'_\Omega) = 1_\Omega$. Define $H_\Phi : \beta\Omega \to \beta\Omega'$ by

$$H_\Phi(\gamma) = \gamma \circ \Phi \quad \text{for all } \gamma \in \beta\Omega.$$

Check that H_Φ is well-defined and continuous.

d) Use b) and c) to show that $\beta\Omega$ satisfies the following universal property:

For any continuous function $h : \Omega \to K$ from Ω into some compact Hausdorff space K, there exists a continuous function $\tilde{h} : \beta\Omega \to K$ such that

$$\tilde{h}(\iota_\omega) = h(\omega) \quad \text{for all } \omega \in \Omega.$$

Comment: If Ω is a Tychonoff space (i.e., if it is completely regular), then it may be shown that the map $\omega \mapsto \iota_\omega$ is a homeomorphism from Ω onto its range $\iota(\Omega)$ (i.e., $\beta\Omega$ is then a compactification of Ω, which is called the Stone-Cech compactification of Ω). You can read more on this in Pedersen's book (cf. 4.3.17 and 4.3.18) if you are interested.

e) Assume that Ω, Ω' are both compact Hausdorff spaces. Show that the following assertions are equivalent:

(i) Ω and Ω' are homeomorphic.

(ii) $C(\Omega)$ and $C(\Omega')$ are isometrically $*$-isomorphic (as C^*-algebras).

(iii) $C(\Omega)$ and $C(\Omega')$ are isomorphic (as algebras).

Exercise 44

Let H be a nontrivial complex Hilbert space and $\mathcal{B} = \{e_j\}_{j \in J}$ be an orthonormal basis for H. Pick $\lambda_j \in \mathbb{C}$ for each $j \in J$, and assume that $\sup_{j \in J} |\lambda_j| < \infty$.

Let $D \in \mathcal{B}(H)$ denote the associated "diagonal" operator satisfying that $D(e_j) = \lambda_j \, e_j$ for every $j \in J$. We recall that $\mathrm{sp}(D) = \overline{\{\lambda_j \mid j \in J\}}$.

a) Check that D is normal.

b) Assume that $f \in C(\mathrm{sp}(D))$. Show that $f(D) \in \mathcal{B}(H)$ is the "diagonal" operator satisfying that $f(D)(e_j) = f(\lambda_j) \, e_j$ for every $j \in J$.